أوراق عمل :

معادلات تفاضلية عادية

ORDINARY DIFFERENTIAL EQUATIONS

اللجنة الأكاديمية لقسم الهندسة الصناعية

2023

LAPLACE TRANSFORMS

1. Which of the following functions are exponential order α?

(1) $f(t) = t^3\sin(t)$ (4) $f(t) = \dfrac{1}{t^2+1}$

(2) $f(t) = \dfrac{t^2}{t+1}$ (5) $f(t) = \sin(t^2)$

(3) $f(t) = e^{\sin(t)}$ (6) $f(t) = 3 - e^{t^2}$

2. Determine whether $f(t)$ is continuous, piecewise continuous, or neither on $[0,\infty)$ and sketch the graph of $f(t)$?

(1) $f(t) = \begin{cases} 1, & 0 \le t \le 1, \\ (t-2)^2, & 1 < t \end{cases}$

(2) $f(t) = \begin{cases} 0, & 0 \le t \le 2, \\ t, & 2 \le t \end{cases}$

(3) $f(t) = \dfrac{t^2-3t+2}{t^2-4}$

3. Compute $\mathscr{L}\{f(t)\}$.

(1) $f(t) = e^{t+1}$ (11) $f(t) = te^{4t}$

(2) $f(t) = e^t\cos(2t)$ (12) $f(t) = t^5$

(3) $f(t) = 4t^2 - 5\sin(3t)$ (13) $f(t) = \sinh(at)$

(4) $f(t) = e^t\cosh(t)$ (14) $f(t) = (e^t - e^{-t})^2$

(5) $f(t) = \cos^2(t)$ (15) $f(t) = \sin(2t)\cos(2t)$

(6) $f(t) = t^2\sin(6t)$ (16) $f(t) = te^{-3t}\cos(3t)$

(7) $f(t) = \sin(4t+5)$ (17) $f(t) = e^{-2t}\cos(4t)$

(8) $f(t) = t^{10}e^{-7t}$ (18) $f(t) = t\left(e^t + e^{2t}\right)^2$

(9) $f(t) = t\mathscr{U}(t-2)$ (19) $f(t) = \cos(2t)\mathscr{U}(t-\pi)$

(10) $f(t) = e^{2-t}\mathscr{U}(t-2)$ (20) $f(t) = (3t+1)\mathscr{U}(t-1)$

4. Compute $\mathscr{L}^{-1}\{F(s)\}$.

(1) $F(s) = \dfrac{1}{2s}$ (13) $F(s) = \dfrac{1}{s^3}$

(2) $F(s) = \dfrac{1}{s^2} - \dfrac{48}{s^5}$ (14) $F(s) = \dfrac{(s+1)^3}{s^4}$

(3) $F(s) = \dfrac{1}{4s+1}$ (15) $F(s) = \dfrac{2s-6}{s^2+9}$

(4) $F(s) = \dfrac{s}{s^2+2s-3}$ (16) $F(s) = \dfrac{s^2+1}{s(s-1)(s-2)}$

(5) $F(s) = \dfrac{6s+3}{s^4+5s^2+4}$ (17) $F(s) = \dfrac{1}{s^4-9}$

(6) $F(s) = \dfrac{1}{s^2-6s+10}$ (18) $F(s) = \dfrac{1}{s^2+2s+5}$

(7) $F(s) = \dfrac{(s+1)^2}{(s+2)^4}$ (19) $F(s) = \dfrac{2s-1}{s^2(s+1)^3}$

(8) $F(s) = \dfrac{e^{-2s}}{s^3}$ (20) $F(s) = \dfrac{se^{-\pi s/2}}{s^2+4}$

(9) $F(s) = \dfrac{e^{-2s}}{s^2(s-1)}$ (21) $F(s) = \dfrac{e^{-\pi s}}{s^2+1}$

(10) $F(s) = \dfrac{e^{-3s}}{s^2+9}$ (22) $F(s) = \dfrac{e^{-2s}}{s^2-3s+5}$

(11) $F(s) = \dfrac{3s-8}{s^2+9}$ (23) $F(s) = \dfrac{4-s}{s^2+2s+10}$

(12) $F(s) = \tan^{-1}\left(\dfrac{1}{s}\right)$ (24) $F(s) = \ln\left(\dfrac{s+1}{s+2}\right)$

5. Compute $\mathscr{L}^{-1}\{F(s)\}$.

(1) $s^2F(s) - 4F(s) = \dfrac{5}{s+1}$

(2) $s^2F(s) + 5F(s) - 6F(s) = \dfrac{s^2+4}{s^2+s}$

6. Solve the given initial value problem using the method of Laplace transforms.

(1) $y'' - 2y' + 5y = 0;\quad y(0) = 2,\quad y'(0) = 4$

(2) $y'' + 6y' + 9y = 0;\quad y(0) = -1,\quad y'(0) = 6$

(3) $y'' + 6y' + 5y = 12e^t;\quad y(0) = -1,\quad y'(0) = 7$

(4) $y'' - y' - 2y = -8\cos(t) - 2\sin(t);$
$y(\frac{\pi}{2}) = 1,\quad y'(\frac{\pi}{2}) = 0$ (**Hint:** Assume that $t = x + \frac{\pi}{2}$)

(5) $y'' - 6y' + 5y = te^t;\quad y(0) = 2,\quad y'(0) = -1$

(6) $y''' - y'' + y' - y = 0;\quad y(0) = 1,\quad y'(0) = 1,\quad y''(0) = 3$

(7) $y'' + y = \mathscr{U}(t-3);\quad y(0) = 0,\quad y'(0) = 1$

(8) $y'' + y = \mathscr{U}(t-2) - \mathscr{U}(t-4);\quad y(0) = 1,\quad y'(0) = 0$

7. Solve the given initial value problem using the method of Laplace transforms.

(1) $y'' + 3y' + 2y = g(t);\quad y(0) = 2,\quad y'(0) = -1$

$$g(t) = \begin{cases} e^{-t}, & 0 \le t < 3, \\ 1, & 3 < t \end{cases}$$

(2) $y'' + 4y = g(t)$ where

$$g(t) = \begin{cases} \sin(t), & 0 \le t \le 2\pi, \\ 1, & 2\pi < t \end{cases}$$

(3) $y'' + 4y = g(t)$ where

$$g(t) = \begin{cases} t, & t < 2, \\ 5, & t > 2 \end{cases}$$

8. Use the definition of the Laplace transform to determine $\mathscr{L}\{f(t)\}$.

(1) $f(t) = \begin{cases} 3, & 0 \le t \le 2, \\ 6-t, & 2 < t \end{cases}$

(2) $f(t) = \begin{cases} e^{-t}, & 0 \le t \le 5, \\ -1, & 5 < t \end{cases}$

(3) $f(t) = \begin{cases} \frac{1}{t}, & 0 < t < 1, \\ 1, & 1 \le t \le 2 \\ 1-t, & 2 < t \le 10 \end{cases}$

LAPLACE TRANSFORMS

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6. Solve the given initial value problem using the method of Laplace transforms.
   (1) $y'' - 2y' + 5y = 0; \quad y(0) = 2, \quad y'(0) = 4$
   (2) $y'' + 6y' + 9y = 0; \quad y(0) = -1, \quad y'(0) = 6$
   (3) $y'' + 6y' + 5y = 12e^t; \quad y(0) = -1, \quad y'(0) = 7$
   (4) $y'' - y' - 2y = -8\cos(t) - 2\sin(t);$
   $y(\frac{\pi}{2}) = 1, \quad y'(\frac{\pi}{2}) = 0$ (**Hint:** Assume that $t = x + \frac{\pi}{2}$)
   (5) $y'' - 6y' + 5y = te^t; \quad y(0) = 2, \quad y'(0) = -1$
   (6) $y''' - y'' + y' - y = 0; \quad y(0) = 1, \quad y'(0) = 1, \quad y''(0) = 3$
   (7) $y'' + y = \mathscr{U}(t-3); \quad y(0) = 0, \quad y'(0) = 1$
   (8) $y'' + y = \mathscr{U}(t-2) - \mathscr{U}(t-4); \quad y(0) = 1, \quad y'(0) = 0$

7. Solve the given initial value problem using the method of Laplace transforms.
   (1) $y'' + 3y' + 2y = g(t); \quad y(0) = 2, \quad y'(0) = -1$
   $$g(t) = \begin{cases} e^{-t}, & 0 \le t < 3, \\ 1, & 3 < t \end{cases}$$
   (2) $y'' + 4y = g(t)$ where
   $$g(t) = \begin{cases} \sin(t), & 0 \le t \le 2\pi, \\ 1, & 2\pi < t \end{cases}$$
   (3) $y'' + 4y = g(t)$ where
   $$g(t) = \begin{cases} t, & t < 2, \\ 5, & t > 2 \end{cases}$$

8. Use the definition of the Laplace transform to determine $\mathscr{L}\{f(t)\}$.
   (1) $f(t) = \begin{cases} 3, & 0 \le t \le 2, \\ 6-t, & 2 < t \end{cases}$
   (2) $f(t) = \begin{cases} e^{-t}, & 0 \le t \le 5, \\ -1, & 5 < t \end{cases}$
   (3) $f(t) = \begin{cases} \frac{1}{t}, & 0 < t < 1, \\ 1, & 1 \le t \le 2 \\ 1-t, & 2 < t \le 10 \end{cases}$

LINEAR HIGHER-ORDER DIFFERENTIAL EQUATIONS

1. If $y_1 = e^{2x}\sin(3x)$ is a solution for $y'' + ay' + by = 0$. Find a and b?

2. Let $y = f(x)$ be a solution for $y'' + x^2y' + e^xy = x - 1$ where $y(0) = 1$, $y'(0) = -1$. Find $f''(0)$?

3. Find a differential equation (DE) satisfied by the family of functions $y(x) = c_1e^{4x} + c_2e^{-4x}$ where c_1 and c_2 are arbitrary constants.

4. Find a DE satisfied by the family of functions $y(x) = c_1x + c_2x^2$ where c_1 and c_2 are arbitrary constants.

5. Find a interval containing $x = 0$ for which the given initial-value problem has unique solution.
   (a) $(x-2)y'' + 3y = x, \quad y(0) = 0, \ y'(0) = 1$
   (b) $y'' + \tan(x)y = e^x, \quad y(0) = 1, \ y'(0) = 0$

6. Verify that $y_1 = x^3$ and $y_2 = |x|^3$ are linearly independent solutions of the DE $x^2y'' - 4xy' + 6y = 0$ on the interval $(-\infty, \infty)$.

7. Determine whether the given functions are linearly independent or linearly dependent on $(-\infty, \infty)$.
   (a) $y_1(x) = e^x$, $y_2(x) = e^{-x}$ and $y_3(x) = \cosh(x)$
   (b) $y_1(x) = e^{x+2}$ and $y_2(x) = e^{x-3}$

8. Use the reduction of order to find a second solution $y_2(x)$ of the given DE's.
   (a) $y'' - 4y' + 4y = 0, \quad y_1 = e^{2x}$
   (b) $y'' - y = 0, \quad y_1 = \cosh(x)$
   (c) $x^2y'' - 7xy' + 16y = 0, \quad y_1 = x^4$
   (d) $x^2y'' - xy' + 2y = 0, \quad y_1 = x\sin(\ln(x))$
   (e) $(1-x^2)y'' + 2xy' = 0, \quad y_1 = 1$
   (f) $(1+x)^2y'' + 2(x+1)y' - 2y = 0, \quad y_1 = x + 1$
   (g) $x^4y^4 - x^3y'' - xy' + 2y = 0, \quad y_1 = x^2e^{\frac{1}{2x^2}}, \quad x > 0$

9. Find the solution of the second order nonlinear equation
$$y'' = -2x(y')^2$$

10. Show that $y_1 = x^3$ and $y_2 = x^4$ forms the fundamental set of solution for $x^2y'' - 6xy' + 12y = 0$

11. If y_1 and y_2 forms the fundamental set of solution for $x^2y'' - 2y' + (x+3)y = 0$.
   (a) Find $W(y_1, y_2)(x)$?
   (b) If $W(y_1, y_2)(2) = 3$. Find $W(y_1, y_2)(5)$?

12. If the Wronskian of any two solutions of the DE
$$y'' + P(x)y' + Q(x)y = 0$$
is constant, what does this imply about the coefficients $P(x)$ and $Q(x)$.

13. Let y_1 and y_2 be two fundamental solutions to the DE
$$y'' + \frac{P(x)}{x}y' + Q(x)y = 0, \quad x > 0.$$
If the Wronskian of any two solutions is x^2e^{-x}, what does this imply about the coefficient $P(x)$.

14. Let $y_1 = 1$, $y_2 = x$, and $y_3(x)$ be solutions of a certain DE with $W(y_1, y_2, y_3)(x) = 2$. Find the general form of $y_3(x)$.

15. Let $y = x^3\ln(x)$ be a solution of the DE
$$Ax^2y'' + Bxy' + cy = 0$$
where A, B and C are constant. Show that $C = 9A$.

16. Find the general solution of the given second-order DE's.
   (a) $ay' + by = 0$,
   (b) $4y'' + y' = 0$
   (c) $y'' - y' - 6y = 0$,
   (d) $y'' + 9y = 0$
   (e) $16y^{(4)} + 24y'' + 9y = 0$,
   (f) $y'' + 8y' + 16y = 0$
   (g) $2y^{(5)} - 7y^{(4)} + 12y''' + 8y'' = 0$
   (h) $y'' + y = 0, \quad y\left(\frac{\pi}{3}\right) = 0, \quad y'\left(\frac{\pi}{3}\right) = 2$
   (i) $y''' + 2y'' - 5y' - 6y = 0, \quad y(0) = y'(0) = 0, \quad y''(0) = 1$

17. Solve the DE $y^{(4)} + y = 0$?

18. Given the auxiliary equation $(r+4)\left(r^2-1\right)^2\left(r^2+4\right) = 0$ for the homogenous DE with constant coefficients. Find the corresponding DE? Find the general solution of the DE?

19. If the root of the auxiliary equation are $r_1 = 4$, $r_2 = r_3 = -5$. Find the corresponding ODE?

20. Find a second-order Cauchy-Euler equation having the solution $y(t) = c_1t + c_2t\ln(t) + \ln(t) + 2$.

21. Solve the given DE's by undetermined coefficients:
   (a) $y'' - 10y' + 25y = 30x + 3$
   (b) $y'' - 8y' + 20y = 100x^2 - 26xe^x$
   (c) $4y'' - 4y' - 3y = \cos(2x)$
   (d) $y^{(4)} - y'' = 4x + 2xe^{-x}$
   (e) $y'' - 4y = (x^2 - x)\sin(2x)$
   (f) $y'' - y' + \frac{1}{4}y = 3 + e^{\frac{x}{2}}$
   (g) $5y'' + y' = -6x, \quad y(0) = 0, \ y'(0) = -10$
   (h) $y'' + y = \sin(x)\cos(2x)$
   Hint: $\sin(A)\cos(B) = \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right)$, $\sin(-x) = -\sin(x)$, $\cos(-x) = \cos(x)$

22. If $r = \pm i$, -1 are the zeros of the equation $r^3 + r^2 + r + 1 = 0$. Find the form of the particular solution of the DE
$$y''' + y'' + y' + y = 5x\sin(x) - 7\sinh(x).$$

Hint: $\sinh(x) = \frac{1}{2}\left(e^x - e^{-x}\right)$, $\cosh(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$

23. Solve the given DE's by variation of parameters:
    (a) $y'' + y = \sec(x)$
    (b) $y'' - y = \sinh(2x)$
    (c) $3y'' - 6y' + 6y = e^x \sec(x)$
    (d) $y'' - 2y' + y = e^x \tan^{-1}(x)$
    (e) $y'' - 4y' + 4y = 6x\,(2x-1)\,e^{2x}, \quad y(0) = 1, \; y'(0) = 0$
    (f) $y''' + 4y' = \sec(2x)$
    (g) $y''' + y' = \tan(x)$

24. Find the values of a, b, c and d so that

$$y(x) = e^{-x} + 2e^{2x} + \sin(4x)$$

is a solution to the DE

$$y'' + ay' + by = c\sin(4x) + d\cos(4x).$$

25. If the DE $y'' + y = -\tan(t)$ has particular solution in the form $y_p = v_1(t)\cos(t) + v_2(t)\sin(t)$, then find $v_2(t)$.

26. Solve the given DE's given that y_1 and y_2 be fundamental solutions to the associated homogenous equation.
    (a) $x^2y'' + xy' + \left(x^2 - \dfrac{1}{4}\right)y = x^{\frac{3}{2}}$ with
    $y_1 = x^{-\frac{1}{2}}\cos(x)$, $y_2 = x^{-\frac{1}{2}}\sin(x)$
    (b) $x^2y'' + xy' + y = \sec(\ln(x))$ with
    $y_1 = \sin(\ln(x))$, $y_2 = \cos(\ln(x))$

27. Solve the DE $xy'' - 3y' = \dfrac{-4}{x}y$.

28. Solve the DE $y'' - 2y' + y = 4x^2 - 3 + x^{-1}e^x$.

29. Solve the DE $x\,(xy')' + 25y = 0$.

30. Transform the DE $at^3y''' + bt^2y'' + cty' + dy = 0$ into a third-order homogeneous linear DE with constant coefficients.
Hint: consider the substitution $t = e^x$

31. Find the homogenous DE with constant coefficients of least degree that has $y(x) = 2e^{3x} - 3\sin(2x) + 2x$ as a solution that satisfies $y(0) = 2$ and $y'(0) = 2$. Find particular solution to $L[y](x) = \cosh(x) + \cos(x)$.

32. Find the general solution of $x^4y'' + x^3y' - 4x^2y = 1$ given that $y_1 = x^2$ is a solution of the associated homogeneous equation.

33. Solve the given DE's:
    (a) $25x^2y'' + 25xy' + y = 0$
    (b) $x^3y''' + xy' - y = 0$
    (c) $x^4y^{(4)} + 6x^3y''' + 9x^2y'' + 3xy' + y = 0$
    (d) $x^2y'' - 4xy' + 6y = \ln(x)$
    (e) $x^3y''' - x^2y'' - 2xy' + 6y = x^2$
    (f) $x^2y'' + xy' + 9y = -\tan(3\ln(x))$
    (g) $x^4y^{(4)} + 6x^3y''' = 0$

34. Find the Cauchy-Euler equation of the second-order having the general solution

$$y(x) = c_1x + c_2x\ln(x) + x\,(\ln(x))^2$$

35. Find the solution to the following DE on any interval not containing $x = -6$.

$$3(x+6)^2y'' + 25(x+6)y' - 16y = 0.$$

36. Let y_1 be a solution for $y'' + x^2y' + y = x^2 - 2x$ and y_2 be a solution for $y'' + x^2y' + y = 5x + 3$. Find the solution of $y'' + x^2y' + y = 2x^2 + 11x + 9$ as linear combination of y_1 and y_2.

37. Assume that $L[y] = xy'' + 2y' - \frac{8}{x}y$. Compute $L[x^3]$.

38. Let $L[y] = a(x)y'' + b(x)y' + c(x)y$. If $e^x + x$, $e^x - 1$, -1 are solutions for $L[y] = g(x)$, $x > 1$, find the general solution of $L[y] = g(x)$.

39. Let $L[y] = 0$ be a linear second-order DE with constant coefficients. If $L[y_1] = 2 - x^2$ and $L[y_2] = x^2 + x + 1$, find the solution to the DE $L[y] = -5x^2 - 3x + 1$.

40. Given the auxiliary equation $(r+7)\,(r^2-1)^2\,(r^2+1) = 0$ for the linear DE with constant coefficients $L[y](x) = 0$. Find the solution to DE $L[y] = e^{-7x}\cos x$

INTRODUCTION

1. Determine the order, unknown function (dependent variable), the independent variable, and linearity of the following equations.

(a) $(y+x)\,y'+y=1$ (d) $(y')^3+y^{(4)}+y=2\sin(t)+\cos^3(t)$ (g) $\dfrac{d^2y}{dx^2}=\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$

(b) $yy'=1$ (e) $u''+(u'+u)'=\cos(r)$ (h) $y'=\dfrac{y+y\cos(xy)}{x+x\cos(xy)}$

(c) $3\dfrac{dy}{dx}+(x+4)\,y=x^2+\dfrac{d^2y}{dx^2}$ (f) $y^{(4)}+\sqrt{x}y'''+\cos(x)=e^{y}$

2. For what value(s) of n will the following equation be linear: $y'-9y^n=t^{2n}\sin(3nt)$

3. Determine all the triplets (m,n,p) such that the differential equation $(y^m-x^2)\,dx+(y^n+x^p)\,dy=0$ is first order linear.

4. Verify that $y(x)=e^{ax}$ is a solution of the initial value problem: $y'=ay$, $y(0)=1$.

5. Verify that $y(x)=10-ce^{-x}$ with c a constant, is a solution to $y'+y=10$.

6. Verify that $x^2+y^2=25$ is an implicit solution of the differential equation $y'=-\frac{x}{y}$ on the open interval $(-5,5)$.

7. Verify that the piecewise-function

$$y=\begin{cases}-x^2, & x<0\\ x^2, & x\geq 0\end{cases}$$

is a solution of the differential equation $xy'-2y=0$ on $(-\infty,\infty)$.

8. Is $y(x)=1$ is a solution of $y''+2y'+y=x$?

9. Show that $y=\ln(x)$ is a solution of $xy''+y'=0$ on $I=(0,\infty)$ but it is not a solution on $I=(-\infty,\infty)$.

10. Find the solution to the IVP: $y'+y=0$, $y(3)=2$, if the general solution to the differential equation is known to be $y(x)=c_1e^{-x}$, where c_1 is an arbitrary constant.

11. Determine c_1 and c_2 so that $y(x)=c_1\sin(2x)+c_2\cos(2x)+1$ will satisfy the conditions $y(\pi/8)=0$ and $y'(\pi/8)=\sqrt{2}$.

12. Write the given differential equations in standard form.

(a) $xy'+y^2=0$ (b) $xy'+\cos(y'+y)=1$ (c) $(x-y)\,dx+y^2\,dy=0$ (d) $dy+dx=0$

13. Suppose that $\varphi(x)$ is a solution of the initial value problem $y''+xy'=x^4$, with $\varphi(-1)=2$, $\varphi'(-1)=6$. Find $\varphi'''(-1)$.

14. If $y=e^{-x}\cos(2x)$ is a solution for $y''+ay'+by=0$, find the values of a and b.

How to find the interval of validity[a]:

- For an initial value problem of a first order linear equation, the interval of validity, if exists, can be found using the following simple procedure. Given $y'+P(x)y=Q(x)$, $y(x_0)=y_0$:
  I. Draw the number line (which is the x-axis).
  II. Find all the discontinuities of $P(x)$ and $Q(x)$. Mark them off on the number line.
  III. Locate on the number line the initial value x_0. Look for the longest interval that contains x_0, but contains no discontinuities.
- For a nonlinear equation, we would need to solve the initial value problem to get the actual interval of validity which could be depend on the value of y_0.

[a]The largest interval where the differential equation is guaranteed to have a unique solution

15. Without solving, determine the interval of validity for the following initial value problem

(a) $(t^2-9)\,y'+2y=\ln|20-4t|$, $y(4)=-3$ (b) $(4-t^2)\,y'+2ty=3t^2$, $y(-3)=1$

(c) $\cos(x)y'=\sin(x)y-\sqrt{x-1}$, $y\left(\frac{3}{2}\right)=0$ (d) $y'+\left(\dfrac{1}{x\ln\left([e^x-3]^2\right)}\right)y=2x$, $y\left(\ln\left(\frac{3}{2}\right)\right)=7$

16. ** Determine the interval of validity for the initial value problem and give its dependence on the value of y_0: $y'=y^2$; $y(0)=y_0$.

FIRST-ORDER DIFFERENTIAL EQUATIONS

1. Classify each differential equation (D.E.) as separable, linear, exact, exact with integrating-factor, homogeneous, Bernoulli, or of the form $y' = G(ax + by)$. Some equations may be more than one kind.

   (a) $x' = 3xt^2$

   (b) $y^{-1}\,dy + ye^{\cos(x)}\sin(x)\,dx = 0$

   (c) $y' - y - e^{3x} = 0$

   (d) $(3 - x^3)\,y' - 3x^2 y = \dfrac{3}{1+x}$

   (e) $\cos\theta\,dr - \left(r\sin\theta - e^{\theta}\right)\,d\theta = 0$

   (f) $(x^2 - xy + y^2)\,dx - xy\,dy = 0$

   (g) $(x + y)\sin(y)\,dx + (x\sin(y) + \cos(y))\,dy = 0$

   (h) $\left(2x + \dfrac{y}{1+x^2y^2}\right)dx + \left(\dfrac{x}{1+x^2y^2} - 2y\right)dy = 0$

   (i) $x' = \sin(x - y)$

   (j) $(3xy - y^2)\,dx + x\,(x - y)\,dy = 0$

   (k) $(x^3 y - 2y^2)\,dx + x^4\,dy = 0$
   (Hint: Find an integrating factor of the form $x^n y^m$)

   (l) $tx' = x^{\frac{1}{2}} - x$

   (m) $\dfrac{dy}{dx} = \dfrac{xy + 3x - y - 3}{xy - 2x + 4y - 8}$

   (n) $\dfrac{dy}{d\theta} = \dfrac{\theta\sec(\frac{y}{\theta}) + y}{\theta}$

   (o) $y' = \sqrt{x + y} - 1$

   (p) $\dfrac{dy}{dx} = \dfrac{x + ye^{\frac{y}{x}}}{xe^{\frac{y}{x}}}$

   (q) $y' + \dfrac{y}{x-2} = 5\,(x - 2)\,y^{\frac{1}{2}}$

   (r) $y' + xy^3 + y = 0$

   (s) $y' = \dfrac{x(y + 1) + (y + 1)^2}{x^2}$

2. Solve the previous D.E's You may leave your answers in implicit form.

3. Solve the following initial-value problems (IVPs):

   (a) $\sqrt{1 - y^2}\,dx - \sqrt{1 - x^2}\,dy = 0, \quad y(0) = \dfrac{\sqrt{3}}{2}$

   (b) $t^2\dfrac{dx}{dt} + 3tx - t^4\ln(t) + 1 = 0, \quad x(1) = 0$

   (c) $\tan(y)\,dx + \left(x\sec^2(y) + \dfrac{1}{y}\right)dy = 0, \quad y(0) = 1$

   (d) $(2x - y)\,dx + (x + y - 3)\,dy = 0, \quad y(0) = 2$

   (e) $\dfrac{dy}{dx} = \dfrac{3x + 2y}{3x + 2y + 2}, \quad y(-1) = -1$

   (f) $y^{\frac{1}{2}}\dfrac{dy}{dx} + y^{\frac{3}{2}} = 1, \quad y(0) = 4$

   (g) $(x^2 + 2y^2)\,dx - xy\,dy = 0, \quad y(-1) = 1$

4. Find a continuous solution of the IVP.
$$\frac{dy}{dx} + 2xy = f(x), \quad y(0) = 2, \quad \text{where}$$
$$f(x) = \begin{cases} x & 0 \le x < 1, \\ 0, & x \ge 1 \end{cases}$$

5. Find the value of k so that $(xy)^{-k}$ is an integrating factor to the D.E.: $y\,dx + (x - 3x^3y^2)\,dy = 0$.

6. Find the value of k so that the following D.E. is exact.
$$(6xy^3 + \cos(y))\,dx + (2kx^2y^2 - x\sin(y))\,dy = 0.$$

7. Show that a one-parameter family of solutions of the equation $(4xy + 3x^2)\,dx + (2y + 2x^2)\,dy = 0$ is $x^3 + 2x^2y + y^2 = C$.

8. Find the function $N(x, y)$ so that the D.E. is exact.
$$\left(x^{-\frac{1}{2}}y^{\frac{1}{2}} + \frac{x}{x^2 + y}\right)dx + N(x, y)\,dy = 0$$

9. Without solving, explain why the IVP $\frac{dy}{dx} = \sqrt{y}$, $y(x_0) = y_0$ has no solution for $y_0 < 0$. Solve the IVP for $y_0 > 0$ and find the largest interval I on which the solution is defined.

10. Find the value of m and n so that the following D.E. is homogenous: $\left(y^2x^2 + \frac{1}{y^m}\right)dx = x^4\left(\ln(y^3) - \ln(x^k)\right)dy$

11. If $f(x)g(y)$ is an integrating factor that makes the D.E. $M(x, y)\,dx + N(x, y)\,dy = 0$ exact. Then show that
$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = \left(\frac{df}{dx}\right)\frac{N}{f} - \left(\frac{dg}{dy}\right)\frac{M}{g}$$

12. Consider the D.E. $M(x, y)\,dx - N(x, y)\,dy = 0$. If $(M_y - N_x) = 2xN$, $(2M_y - N_x) = 2yM$, and $(M_y - 2N_x) = M$. Find the integrating factor to the D.E..

13. Consider the D.E. $M(x, y)\,dx + N(x, y)\,dy = 0$. If $(M_y)^2 - (N_x)^2 = \cos(x)\,(M_y + N_x)\,N$, and $\cos(x) > 0$, $(M_y + N_x) \neq 0$. Find the integrating factor to the D.E..

14. If $\dfrac{M(x, y) - N(x, y)}{N(x, y)} = x$. Find the general solution to the D.E. $\dfrac{M}{x}\,dx - \dfrac{N}{y}\,dy = 0$, $x, y > 0$.

15. Solve the following D.Es

    (a) $3x^3y^2y' = \sec^2(x^2y^3) - 3x^2y^3 + 1$.
    (Hint: Use the substitution $v = x^3y^3$)

    (b) $ye^{xy}\dfrac{dx}{dy} + xe^{xy} = 12y^2$, $x, y > 0$.
    (Hint: Use the substitution $v = e^{xy}$)