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دفتّر :

معادلات تفاضلية عادية

Ordinary Differential Equations

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Def: Equations contain derivatives on a dependent variable with respect to the independent variable are called **Differential equation (DE)**.

For example: $y'' - y = x^2$

Here $\Rightarrow$ x is independent variable

y is dependent variable

Unknown: $y = y(x)$

The equation $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3x$

Here $\Rightarrow$ (x, y) is independent variables

u is dependent variable

$u = u(x, y)$

Classifications of DEs

(1) Order: The order of a DE is the order of the highest derivative appear on the dependent variable with respect to the independent variable.

For example: $y'' - y = x^3$ **2nd Order DE**

$(y')^2 + y^4 = \sin x$ **1st Order DE**

$y^{(4)} + y'' = x^{10}$ **4th Order DE**

(2) Linear and Non Linear Equation: A DE is called linear if it's linear of y and all its derivatives ($y', y'', \dots$) if the DE is not linear, we call it Non Linear.

For example :

$$y'' + 3y = x^3$$

Linear - 2nd Order DE

$$y' + y^2 = 3x$$

Non Linear - 1st Order DE

$$yy'' + y' = \ln x$$

Non Linear - 2nd Order DE

$$y'' + x^2y = 3$$

Linear - 2nd Order DE

(3) Ordinary and Partial DEs :

Ordinary : if all the derivatives on the dependent variable are ordinary derivatives the DE called (ODE)

as $y = y(x)$ y depends on only one variable.

Partial : if we have Partial derivatives on the DE then we call it (PDE) Here $u = u(x_1, x_2, \dots)$

$$\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots$$

For example :

$$y'' + y' - 2y = x$$

2nd Order - Linear ODE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$u = u(x, y)$ PDE

Solutions of DEs : in general an (ODE) of order (n) can be written as

$$y^{(n)} = f(y, y', y'', \dots, y^{(n-1)})$$

$\phi(x)$ is a solution to this (ODE) if $(\phi, \phi', \phi'', \dots, \phi^{(n)})$ exist and ϕ satisfy the (ODE) $\phi^{(n)} = f(\phi, \phi', \phi'', \dots, \phi^{(n-1)})$

Example 8 ($Y'' + Y = 0$) claim: $y(x) = \cos(x)$ is a solution.

Verify: $y(x) = \cos(x) \Rightarrow y'(x) = -\sin(x) \Rightarrow y''(x) = -\cos(x)$

$$Y'' + Y = -\cos(x) + \cos(x) = 0 \quad \checkmark$$

So $y(x) = \cos(x)$ is a solution to the DE.

Example 8 ($Y' + Y = 0$) claim: $y(x) = e^{-x}$ is a solution.

$$Y' = -e^{-x} \Rightarrow Y' + Y = -e^{-x} + e^{-x} = 0$$

So $y(x) = e^{-x}$ is a solution to the DE.

EX 8 show that $Y - \ln Y = x^2 + 1$ defines an implicit solution to the DE $\Rightarrow Y' = \frac{2xy}{Y-1}$ (nonlinear, first order)

Using Imp differentiation:

$$(Y - \ln Y)' = (x^2 + 1)'$$

$$Y' - \frac{Y'}{Y} = 2x \Rightarrow Y' \left(Y - 1 \right) = 2xY \Rightarrow Y' = \frac{2xY}{Y-1} \quad \#$$

EX 8 Find the values of r such that $y(x) = e^{rx}$ is a solution to the DE $y'' + 5y' + 6y = 0$

$$y(x) = e^{rx}$$

$$y'(x) = r e^{rx}$$

$$y''(x) = r^2 e^{rx}$$

$$r^2 e^{rx} + 5r e^{rx} + 6e^{rx} = 0$$

$$e^{rx} (r^2 + 5r + 6) = 0$$

$$e^{rx} \neq 0$$

$$r^2 + 5r + 6 = 0$$

$$(r+3)(r+2)$$

$$r = -2, -3$$

$$y(x) = e^{-2x}, y(x) = e^{-3x} \text{ are solutions.}$$

Initial Value Problems

$$y^{(n)} = f(x, y, y')$$

along with initial conditions (IC) :

$$y(x_0) = \alpha_0, \quad y'(x_0) = \alpha_1, \quad y''(x_0) = \alpha_2$$

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example 8 $y' + 2y = x^2$ DE $\rightarrow$ (IVP)
 $y(1) = 3$ IC

المعادلة

DE (1) $y' + 2y = x^2$

IC (2) $y(1) = 3$

example 8 $y(x) = e^x$ is a solution to the IVP

$$y' - y = 0$$

$$y(0) = 1$$

$$y' = e^x, \quad y = e^x$$

its a solution.

$$e^x - e^x = 0 \quad \checkmark$$

$$e^0 - e^0 = 0 \quad \checkmark$$

example 8 Given that $y(x) = C_1 e^x + C_2 e^{-x}$ is a solution to the DE $\Rightarrow y'' - y = 0$ for any choice of C_1, C_2 . Find C_1, C_2 so that the IC's satisfied $y(0) = 2, y'(0) = 1$.

General Solution

$$y' = C_1 e^x - C_2 e^{-x}$$

$$y'' = C_1 e^x + C_2 e^{-x}$$

$$C_1 e^x - C_2 e^{-x} + C_1 e^x + C_2 e^{-x} = 0$$

$$y(0) = 2 \Rightarrow (C_1 + C_2 = 2) \quad (1)$$

$$y'(0) = 1 \Rightarrow (C_1 - C_2 = 1) \quad (2)^+$$

$$2C_1 = 3 \Rightarrow C_1 = 3/2$$

$$\frac{3}{2} + C_2 = 2 \Rightarrow C_2 = 1/2$$

So $y(x) = \frac{3}{2} e^x + \frac{1}{2} e^{-x} \Rightarrow$ IVP

Particular Solution

Note 8. General Solutions

يعني اي حل لهذه المعادلة يسمى حل عام لاننا نكتب على كل صورة نفس النظرية الطريقة المستعملة لا يحدد.

- particular Solution

حل خاص لهذه المعادلة.

Another note : Constant وعدم $\frac{dy}{dx}$ يكون في DE لان $\frac{dy}{dx}$ لا يمكن ان يكون صفر، $Order$ لان $\frac{dy}{dx}$ لا يمكن ان يكون صفر، DE لان $\frac{dy}{dx}$ لا يمكن ان يكون صفر، DE لان $\frac{dy}{dx}$ لا يمكن ان يكون صفر.

Chapter (2) : First Order ODEs

The general form of a First Order ODE is $(y' = f(x, y))$

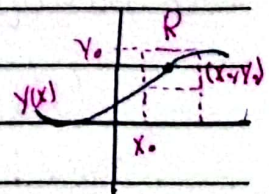
A linear first order (ODE) has the form $(a_1(x)y' + a_2(x)y = a_3(x))$

For example : $y' + y^2 = \sin x$ non linear 1st order (ODE)

$y' + x^2 y = x^{10}$ linear 1st order (ODE)

Theorem 8 Consider the IVP $y' = f(x, y)$

$$y(x_0) = y_0$$



Let (R) be the rectangle contains (x_0, y_0)

(a) IF $f(x, y)$ continuous in R , then the (IVP) has a solution on an open interval $(x_0 - \delta, x_0 + \delta)$.

(b) If in addition $\frac{dy}{dx}$ continuous on $\mathbb{R}$ then the (IVP) has a Unique Solution on an interval $(x_0 - \delta, x_0 + \delta)$

Forexample 8 $y' - 2xy = x^3$

$$y' = x^3 + 2xy$$

$$y' = f(x, y)$$

$$f(x, y) = x^3 + 2xy \text{ cts}$$

$$\frac{df}{dy} = 2x \text{ cts (exist, has a Unique solution).}$$

Linear First order (ODEs)

the general form of a First order ODE is $a_1(x)y' + a_2(x)y = a_3(x)$

- Divide by $a_1(x)$ to get: $y' + \underbrace{\left(\frac{a_2(x)}{a_1(x)}\right)}_{p(x)} y = \underbrace{\left(\frac{a_3(x)}{a_1(x)}\right)}_{q(x)}$

$$y' + p(x)y = q(x) \quad \text{--- (1)}$$

So this is called the standard form of a first order linear ODE.

Now let $\mu(x)$ be a function to be determined.

Multiply Eq. (1) by $\mu(x)$

$$\mu(x)y' + p(x)\mu(x)y = \mu(x)q(x)$$

This eqn is equivalent to $\mu(x)y' + \mu(x)p(x)y = \mu(x)q(x)$

$$(\mu(x)y)' + (\mu(x)p(x) - \mu'(x))y = \mu(x)q(x)$$

We choose $\mu(x)$ such that $(\mu(x)p(x) - \mu'(x)) = 0$

$$\text{Now: } \int \frac{\mu'(x)}{\mu(x)} dx = \int p(x) dx + C$$

$$\ln \mu(x) = \int p(x) dx$$

$$\mu(x) = e^{\int p(x) dx}$$

choosing $\mu(x) = e^{\int p(x) dx}$, reduce the test equation to $(\mu(x)y)' = \mu(x)g$

Now $\mu(x)y = \int \mu(x)g(x) dx + C$

$$y(x) = \frac{\int \mu(x)g(x) dx + C}{\mu(x)}$$

Summary :

To solve a Linear first Order (ODE) follow the steps:

- (1) Make Sure the (DE) is in standard form $(y' + p(x)y = g(x))$
- (2) Determine $p(x)$ and $g(x)$
- (3) Compute the integration factor $\mu(x) = e^{\int p(x) dx}$
- (4) The general Solution is $y(x) = \frac{\int \mu(x)g(x) dx + C}{\mu(x)}$

(Problems) :

Solve the DE $(x^2+1)y' + xy = x$ (first order, linear)

We first put it at its standard form $y' + \frac{x}{x^2+1}y = \frac{x}{x^2+1}$

$$\mu(x) = e^{\int \frac{x}{x^2+1} dx} = e^{\frac{1}{2} \ln \frac{x^2}{x^2+1}} = e^{\frac{1}{2} \ln \frac{x^2}{x^2+1}} = (x^2+1)^{1/2}$$

$$\text{The general Solution } y(x) = \frac{\int (x^2+1)^{1/2} \left(\frac{x}{x^2+1} \right) dx + C}{(x^2+1)^{1/2}}$$

$$y(x) = \frac{(x^2+1)^{1/2} + C}{(x^2+1)^{1/2}} = 1 + (x^2+1)^{-1/2}C$$

substitution $u = x^2+1 \quad dx = \frac{du}{2x}$ $\int x \cdot u^{-1/2} \frac{du}{2x} = \frac{1}{2} \int u^{-1/2} du = (x^2+1)^{1/2}$	
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Solve the IVP $xy' + 2y = \sin(x)$, $y(\pi/2) = 1$ (Linear, first order)

$$y' + \left(\frac{2}{x}\right)y = \left(\frac{\sin(x)}{x}\right) \rightarrow q(x)$$

$$\mu(x) = e^{\int \frac{2}{x} dx} = e^{2 \ln x} = x^2$$

$$y(x) = \int x^2 \frac{\sin(x)}{x^2} dx + C$$

$$y(x) = -x \cos(x) + \sin(x) + C$$

$$y(x) \Big|_{x=\pi/2} = -\frac{\pi}{2} \cos(\pi/2) + \sin(\pi/2) + C = 1$$

$$x = \pi/2$$

$$y = 1$$

$$\frac{\pi^2}{4}$$

$$(C = 1 - \frac{\pi^2}{4})$$

the general solution $\Rightarrow y(x) = \frac{-x \cos(x) + \sin(x) + \frac{\pi^2}{4} - 1}{x^2}$

Solve the DE by regarding (y) as independent variable rather than (x)?

$$(1 + 2xy) \frac{dy}{dx} = 1 + y^2$$

$$\frac{dx}{dy} = \frac{1 + 2xy}{1 + y^2}$$

$$= \frac{1}{1 + y^2} + \frac{2xy}{1 + y^2}$$

$$\left(x\right)' + \left(\frac{dx}{dy}\right) \left(\frac{2x}{1+y^2}\right) = \left(\frac{1}{1+y^2}\right) \int q(y)$$

$$\mu(y) = e^{\int \frac{-2y}{1+y^2} dy} = e^{-\ln(1+y^2)} = (1+y^2)^{-1}$$

$$x(y) = \dots$$

التكامل بالجزأ 8

واحد من أساليب التكامل وحل المسألة

أي شيء اشتق منه ما هو التكامل أي شيء اشتق منه

(-) هو التكامل الذي يعني التكامل (x) مشتقة

أي شيء اشتق منه

Find the largest interval in which the solution the IVP exist.

$$xy' + \tan(x)y = \sin(x)$$

$$y(\pi) = 6$$

$$y' + \frac{\tan(x)}{x} y = \frac{\sin(x)}{x}$$

$$y' = f(x, y) \quad y(x_0) = y_0$$

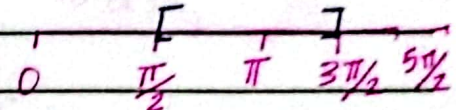
$$f(x, y) = \frac{\sin(x)}{x} - \frac{\sin(x)}{x \cos(x)} y$$

المستطابق على $(0, \pi/2, 3\pi/2, 5\pi/2, \dots)$

المستطابق على $(0, \pi/2, 3\pi/2, 5\pi/2, \dots)$

والاستمرارية

$$\frac{df}{dy} = -\frac{\sin(x)}{x \cos(x)}$$



the largest interval $(\pi/2, 3\pi/2)$

كلما كانت x قريبة من $\pi/2$ أو $3\pi/2$ تكون $f(x, y)$ قريبة من $\pm\infty$ فالحل لا يمكن أن يمتد إلى هذه النقاط.
The solution cannot be extended to these points.

Bernoulli's equation

The first order (ODE) of the form

$$y' + P(x)y = Q(x)y^n \quad \text{as } n \neq 0, 1$$

To solve Bernoulli equation

Use the Substitution $(v = y^{1-n})$

$$\text{Differentiate } v = (1-n)y^{-n} y'$$

multiply the Bernoulli equation by $(1-n)y^{-n}$

$$(1-n)y^{-n}y' + (1-n)p(x)y^{-n}y = (1-n)q(x)y^{-n}y^n$$

$$v' + (1-n)p(x)v = (1-n)q(x)$$

became linear ODE of v solve it to find v then use

$$(v = y^{1-n}) \text{ to find } y.$$

example 8 Solve the DE $y' = e^{x+2y}$

$$y' = e^x e^{2y}$$

$$\frac{dy}{dx} = e^x e^{2y}$$

$$\int e^{-2y} dy = \int e^x dx$$

$$\left(\frac{e^{-2y}}{-2} = e^x + C \right) \Rightarrow \text{implicit solution}$$

First Order :

Linear

Bernoulli

Separable

Homogeneous

Exact

Special
integration
facts

Homogeneous Equation :

if the DE $y' = F(x, y)$ Can Be Written in the form :

$$y' = G\left(\frac{y}{x}\right)$$

then the DE is called homogeneous.

Example :

$$(1) \quad y' = \frac{x-3y}{2x+y} \stackrel{(\div x)}{=} \frac{1-3\left(\frac{y}{x}\right)}{2+\left(\frac{y}{x}\right)} \Rightarrow \text{let } v = \frac{y}{x} \text{ \& } y' = \frac{1-3v}{2+v} = G(v) = G\left(\frac{y}{x}\right)$$

$$(2) \quad y' = \frac{y^2 + x \sqrt{x^2 + y^2}}{xy} \stackrel{(\div x^2)}{=} \left(\frac{y}{x}\right)^2 + \frac{\sqrt{x^2 + y^2}}{x} = \left(\frac{y}{x}\right)^2 + \sqrt{\frac{x^2 + y^2}{x^2}}$$

$$= \frac{v^2 + \sqrt{1+v^2}}{v}$$

Consider the homogeneous DE :

$$y' = G\left(\frac{y}{x}\right)$$

To solve it, Use the Substitution : $v = \frac{y}{x} \Rightarrow x v = y$

Differentiate to get $y' = x v' + v$ then substituting in the DE.

$$\begin{aligned} y' &= G\left(\frac{y}{x}\right) \\ x v' + v &= G(v) \\ x \frac{dv}{dx} &= G(v) - v \end{aligned}$$

$$\frac{1}{G(v) - v} dv = \frac{1}{x} dx \Rightarrow \text{A Separable DE solve it for } v \text{ the substitution}$$

$$v = \frac{y}{x}$$

example : Solve the DE $y' = \frac{y^2 - 3x^2}{xy - x^3 y^4}$

$$y' = \frac{\left(\frac{y}{x}\right)^2 - 3}{\left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^4} = G\left(\frac{y}{x}\right) \text{ (Hom. DE)}$$

$$v = \frac{y}{x} \Rightarrow y = x v \Rightarrow y' = x v' + v$$

$$\frac{v^2 - 3}{v - v^4} = x v' + v$$

$$x \frac{dv}{dx} = \frac{v^2 - 3}{v - v^4} - v = \frac{(v^2 - 3) - (v^2 - v^4)}{v - v^4}$$

$$x \frac{dv}{dx} = \frac{-2}{v^3 - 1} = -\frac{2v}{v^3 - 1}$$

$$\int \frac{v^2 - 1}{v} dv = \int -\frac{2}{x} dx$$

$$\frac{v^2}{2} - \ln v = -2 \ln x + C \Rightarrow \frac{y^2}{2x^2} - \ln\left(\frac{y}{x}\right) = -2 \ln x + C$$

implicit solution

example: $\frac{dy}{dx} = \frac{y(\ln y - \ln x + 1)}{x}$

$$\frac{dy}{dx} = \frac{y}{x} \left(\ln \frac{y}{x} + 1 \right)$$

$$xv' + v = v(\ln v + 1)$$

$$x \frac{dv}{dx} = v \ln v + v$$

$$u = \ln v$$

$$du = \frac{1}{v} dv$$

$$dv = u du$$

$$\int \frac{dv}{v \ln v} = \int \frac{1}{x} dx$$

(المتكامل)

$$= \ln x + C$$

Exact Equation

Any first order ODE $y' = f(x, y)$ Can Be written in the Differential

Form: $M(x, y)dx + N(x, y)dy = 0$

For example: $y' = \frac{2x^2y + 3x - 2}{xy^3 + e^{xy}}$

$$\frac{dy}{dx} = \frac{2x^2y + 3x - 2}{xy^3 + e^{xy}}$$

$$(2 - 2x^2y - 3x)dx = (xy^3 + e^{xy})dy = 0$$

example: Consider the DE $(2x + y^2)dx + (2xy)dy = 0$

Function
of (x, y)

$$\frac{d}{dx} (x^2 + xy) + \frac{d}{dy} (x^2 + xy^2) dy = 0$$

$C = \text{constant}$

$$\frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = 0 \Rightarrow \partial F = 0 \text{ (total derivative)}$$

$$F(x, y) = C \text{ is a solution to the DE } (x^2 + xy^2 = C)$$

implicit solution

The DE $M(x,y) dx + N(x,y) dy = 0$ is called exact if there is a function $F(x,y)$ such that $\frac{\partial F}{\partial x} = M(x,y)$, $\frac{\partial F}{\partial y} = N(x,y)$

Theorem: The DE $M(x,y) dx + N(x,y) dy = 0$ is exact if and only if $(\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x})$ (condition for exactness)

Example: $M(x,y) = 2x + y^2$
 $N(x,y) = 2xy$

Here $\Rightarrow \frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial x} = 2y$ Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ the DE is exact.

if the DE $M(x,y) dx + N(x,y) dy = 0$ is exact, then there is a function $F(x,y)$ such that $\frac{\partial F}{\partial x} = M(x,y)$, $\frac{\partial F}{\partial y} = N(x,y)$ if we find such $F(x,y)$ then $F(x,y) = C$ is an implicit solution to the DE.

example: Consider the DE $(2x + y^2) dx + (2xy) dy = 0$

Here: $M(x,y) = 2x + y^2$, $N(x,y) = 2xy$

(1) Check the exactness: $\frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial x} = 2y$ exact ✓.

(2) Thus there is a function $F(x,y)$ satisfying $\frac{\partial F}{\partial x} = M(x,y) = 2x + y^2$
 $\frac{\partial F}{\partial y} = N(x,y) = 2xy$

(3) We need to find such $F(x,y)$ using that:

$$\frac{\partial F}{\partial x} = 2x + y^2 \Rightarrow F = \int (2x + y^2) dx = x^2 + y^2 x + h(y)$$

(4) To Determine $F(x,y)$ we need to find $h(y)$ using that:

$$\frac{\partial F}{\partial y} = 2xy \Rightarrow 2yx + h'(y) = 2xy$$

$$h'(y) = 0 \Rightarrow h(y) = C$$

then $F(x,y) = x^2 + xy^2 = C$ is an implicit solution

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example 8 Solve the DE $(ye^{xy} \tan(x) + e^{xy} \sec^2(x))dx + (xe^{xy} \tan(x))dy = 0$

(1) exactness test $\Rightarrow \frac{\partial M}{\partial y} = (xy e^{xy} + e^{xy}) \tan(x) + x \sec^2(x) e^{xy}$
 $\frac{\partial N}{\partial x} = xe^{xy} \sec^2(x) + \tan(x) (xy e^{xy} + e^{xy})$ ✓ (exact)

• $\frac{\partial F}{\partial x} = ye^{xy} \tan(x) + e^{xy} \sec^2(x)$

• $\frac{\partial F}{\partial y} = xe^{xy} \tan(x)$

• $F = \int xe^{xy} \tan(x) dy$

$-x \tan(x) \frac{e^{xy}}{-x} + h(x) \Rightarrow e^{xy} \tan(x) + h(x)$

• We need to find $h(x)$?

$\frac{\partial F}{\partial x} = ye^{xy} \tan(x) + e^{xy} \sec^2(x)$

$e^{xy} \sec^2(x) + \tan(x)' y e^{xy} + h'(x) = ye^{xy} \tan(x) + e^{xy} \sec^2(x)$

$h'(x) = 0$ then $h(x) = C$

So $F(x,y) = e^{xy} \tan(x) + C$ But $F(x,y) = C$ then $(e^{xy} \tan(x) - C)$ implicit solution.

example 8 Find the most general function $M(x,y)$ so that the DE is exact. $M(x,y) + (\sin(x) \cos(y) - xy - e^{-y}) dy = 0$

Since the DE is exact then $(\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x})$

$\frac{\partial M}{\partial y} = \cos(x) \cos(y) - y - 0$

$M = \int \cos(x) \cos(y) - y dy$

$M = \cos(x) \sin(y) - \frac{y^2}{2} + h(x)$ // general

Special integration factor :

Consider the DE $(2y-6x)dx + (3x-4x^2y^{-1})dy = 0$.

• Check the exactness

$$\frac{\partial M}{\partial y} \stackrel{?}{=} \frac{\partial N}{\partial x} \Rightarrow 2 \neq 3-8xy^{-1} \quad \text{Since } \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \text{ the DE isn't exact.}$$

• multiply the DE By the function $(U = xy^2)$

$$(2xy^3 - 6x^2y^2)dx + (3x^2y^2 - 4x^3y)dy = 0$$

$$\begin{array}{l} M(x,y) \\ \frac{\partial M}{\partial y} = 6xy^2 - 12x^2y \end{array} \quad \begin{array}{l} N(x,y) \\ \frac{\partial N}{\partial x} = 6xy^2 - 12x^2y \end{array}$$

• After multiplying the DE (xy) the DE Became exact. (integrating factor).

• Consider the non exact DE $M(x,y)dx + N(x,y)dy = 0$ the function $U(x,y)$ is called an integrating factor to the DE When we multiply the DE By $U(x,y)$ the DE Became exact.

let $U(x,y)$ Be integrating factor to $M(x,y)dx + N(x,y)dy = 0$
Then $U(x,y)M(x,y)dx + U(x,y)N(x,y)dy = 0$ is exact.

$$\frac{d}{dy} (\underbrace{U}_{\text{integrating factor}} \underbrace{M}_{\text{coefficient of } dx}) = \frac{\partial}{\partial x} (\underbrace{U}_{\text{integrating factor}} \underbrace{N}_{\text{coefficient of } dy})$$

$$(U \frac{\partial M}{\partial y} + \frac{\partial U}{\partial y} M) = U \frac{\partial N}{\partial x} + \frac{\partial U}{\partial x} N$$

let's assume that $U = U(x)$ (function of x only)

$$N \frac{\partial U}{\partial x} = (\frac{\partial U}{\partial y} M - \frac{\partial U}{\partial x} N)$$

$$\int \frac{U'}{N} = \int \frac{\frac{\partial U}{\partial y} M - \frac{\partial U}{\partial x} N}{N} \quad \text{if } U(x) \text{ is a function of } x \text{ only}$$

$$U(x) = e^{\int (\frac{\frac{\partial U}{\partial y} M}{N} - \frac{\partial U}{\partial x}) dx}$$

On the other side, assume that $\mu = \mu(y)$ (بافتراض أن μ دالة في y فقط)

$$\mu \frac{\partial M}{\partial y} + \mu \frac{\partial M}{\partial y} = \mu \frac{\partial N}{\partial x} + \frac{\partial M}{\partial x} N$$

$$\mu \frac{\partial M}{\partial y} + \mu \frac{\partial M}{\partial y} = \mu \frac{\partial N}{\partial x} \Rightarrow \mu' = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

if $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$ depend on y only, then $\mu(y) = e^{\int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dy}$

example: Find an integrating factor to the DE:

$$(2xy^3 - 2x^3y^3 - 4x^2y + 2x)dx + (3x^2y^2 + 4y)dy = 0$$

$M(x,y)$ $N(x,y)$

• we first try to find $\mu(x) = \mu$ (Not exact)

$$\frac{\partial M}{\partial y} = 6xy^2 - 6x^3y^2 - 4x^2$$

$$\frac{\partial N}{\partial x} = 6xy^2$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{6xy^2 - 6x^3y^2 - 8xy - 6xy^2}{3x^2y^2 + 4y} \quad \text{مستطاع } (-2xy)$$

$$= \frac{-2xy(3x^2y + 4)}{y(3x^2 + 4)} = (-2x) \text{ depend on } x \text{ only}$$

$$\text{So } \mu(x) = e^{\int -2x dx} = e^{-x^2} \text{ an integrating factor.}$$

example 8 $(2xy^3) dx + (3x^2y^2 + x^2y^3 + 1) dy = 0$

$\left(\frac{\partial M}{\partial y} = 6xy^2, \frac{\partial N}{\partial x} = 6xy^2 + 2xy^3 \right)$ not exact.

we try to find $u = u(x)$

$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{6xy^2 - 6xy^2 - 2xy^3}{3x^2y^2 + x^2y^3 + 1}$ (مقسوم) (مقسوم عليه) Doesn't depend on (x) only

we next try $u = u(y)$

$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{2xy^3 + 6xy^2 - 6xy^2}{2xy^3} = 1$ So it's depend on (y) only.

$u = \int 1 dy = e^y$ # an integrating factor.

example 8 Solve the DE by finding an appropriate integrating factor:

$(y+1) dx + (2x - e^y) dy = 0$
 $M(x,y) \quad N(x,y)$

• we try first to find $u = u(x)$

$\frac{\partial M}{\partial y} = 1, \frac{\partial N}{\partial x} = 2$

$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{1 - 2}{2x - e^y}$ Doesn't depend on (x) only.

$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{2 - 1}{y + 1} = \frac{1}{y + 1}$ depend on (y) only.

$$M(y) = \int \frac{1}{y+1} dy = e^{\ln(y+1)} = (y+1)$$

• Now multiply the DE By $(y+1)$

$$(y+1)^2 dx + (y+1)(2x - e^y) dy = 0$$

$(y^2 + 2y + 1) dx + (2xy - ye^y + 2x - e^y) dy = 0 \rightarrow$ this DE is exact so there is a function:

$F(x,y)$ satisfies $(\frac{\partial F}{\partial x} = M, \frac{\partial F}{\partial y} = N) \Rightarrow$ (exact = yes)

$$\text{Using: } \frac{\partial F}{\partial x} = y^2 + 2y + 1 \Rightarrow F = \int y^2 + 2y + 1 dx$$

$$F = xy^2 + 2yx + x + h(y)$$

To determine $h(y)$ we use

$$\frac{\partial F}{\partial y} = 2xy + 2x - ye^y - e^y$$

$$2yx' + 2x' + h'(y) = 2xy + 2x - ye^y - e^y$$

$$\int h'(y) = \int ye^y - e^y dy = -\int e^y (y+1) dy = -(y+1)e^y - e^y$$

$$h(y) = e^y - (y+1)e^y = (-ye^y)$$

• So $F(x,y)$ is completely determined $F(x,y) = xy^2 + 2xy + x - ye^y$ then $F(x,y) = C$

$xy^2 + 2xy + x - ye^y = C$ is an implicit solution to the DE.

Substitution and Transformations

Example 8 Solv ($y' = \sqrt{x+y} - 1$) $\Rightarrow$ non linear, not separable, not Bernoulli, not Homogeneous.

Let $v = x + y$

$v' = 1 + y'$

Substituting in the DE $\Rightarrow v' - 1 = v^{1/2} - 1$

$\frac{\partial v}{\partial x} = v^{1/2} \Rightarrow$ Became separable.

So $\int v^{-1/2} dv = \int dx$

$2v^{1/2} = x + C \Rightarrow (2(x+y))^{1/2} = x + C$ implicit solution.

Example 8 Use the substitution $y = vx^2$ to solve

$\frac{dy}{dx} = \frac{2y}{x} + \cos\left(\frac{y}{x^2}\right) \Rightarrow$ not linear, not separable, not Bernoulli, not Homogeneous.

Use ($y = vx^2$)

$(v = \frac{y}{x^2}) \xrightarrow{\text{Diff}} \frac{\partial y}{\partial x} \Rightarrow (2x)v + x^2 v'$

Substituting in the DE $\Rightarrow 2xv + x^2 \frac{\partial v}{\partial x} = \frac{2y}{x} + \cos(v)$

$2xv + x^2 \frac{\partial v}{\partial x} = 2xv + \cos(v)$

$x^2 \frac{dv}{dx} = \cos(v)$ separable

$\int \frac{dv}{\cos(v)} = \int \frac{1}{x^2} dx$

$\int \sec(v) dv = -x^{-1} + C$

$|\ln|\sec(v) + \tan(v)|| = -x^{-1} + C$

$(\ln|\sec(\frac{y}{x^2}) + \tan(\frac{y}{x^2})| = -x^{-1} + C)$ implicit solution.

Example 8 Solve the DE: $2xe^{2y} \frac{dy}{dx} = 3x^4 + e^{2y}$

Let $v = e^{2y}$

$$\left(\frac{dv}{dx} = 2e^{2y} y' \right) \times (x)$$

بلا خط انما بتقسيمه بحاصل سريعه المعادلة
لكي يبينه فيها (x) بالتالي د جنس الطرفين
ب (x)

$$x \frac{dv}{dx} = 2xe^{2y} y'$$

$$x \frac{dv}{dx} = 3x^4 + v \quad (\div x) \Rightarrow v' = 3x^3 + \frac{v}{x} \quad (\text{became linear eqn})$$

$$\text{So } v' - \frac{1}{x} v = 3x^3$$

$$P(x) = -1/x, \quad Q(x) = 3x^3$$

$$I(x) = e^{\int 1/x dx} = x^{-1}, \quad Y(x) = \int x^{-1} \cdot 3x^3 + C = x^3 + C - (x^4 + Cx)$$

Example 8 Solve $y^2 \frac{dy}{dx} + y^3 = 4e^x$

سأفعل غير قسري

Bernolli

$$\frac{dy}{dx} + y = 4y^{-2}e^x$$

اذا ضربنا الـ (y^2) يتجمع

كلها Linear

Let $v = y^3$

$$v' = 3y^2 y' \Rightarrow 3y^2 y' + 3y^3 = 12e^x$$

ملاحظة

$$v' + 3v = 12e^x$$

مما كانت المعادلة

$$P(x) = 3, \quad Q(x) = 12e^x$$

Bernolli

$$I(x) = e^{\int 3 dx} = e^{3x}$$

(Linear) لا خطها

$$y(x) = \int 12e^{3x} e^x dx + C (\dots)$$

سید (Hama)

$$y' = \frac{y}{x} + \sqrt{1 - \left(\frac{y}{x}\right)^2} \quad \text{let } v = \frac{y}{x}, \quad y = xv, \quad y' = x \frac{dv}{dx} + v$$

Became seproble.

$$\sin^{-1}(v) = \ln(x) + C$$

~~$$\sin^{-1}\left(\frac{y}{x}\right) = \ln(x) + C$$~~

Example 8 Solve the DE: $(1 + \frac{2x}{y}) dx + (2 - \frac{x^2}{y^2}) dy = 0$

$$\frac{\partial U}{\partial y} = \frac{\partial U}{\partial x}$$

$$\frac{-2x}{y^2} = \frac{-2x}{y^2} \quad (\text{exact}) \checkmark$$

then $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$ (...)

خطة العمل : عرضة نوع المعادلة :

- Linear $\Rightarrow$ مستقيم لا أس
- Seprable $\Rightarrow$ قابل لفك (عوامل منفصلة)
- Bernolli $\Rightarrow$ بنبرلي (الخطية) والفرعية y^n
- Homogeneous $\Rightarrow$ متجانس $\frac{f(x,y)}{f(x,y)}$
- Exact $\Rightarrow$ من شكل المطابقة
- integrating factor
- Substitution. $\Rightarrow$ إحتراس من الجأ إليه

Chapter (3) Linear 2nd Order ODEs.

A general form of a 2nd Order ODE is

$$y'' = f(x, y, y')$$

A linear 2nd Order ODE has the form

$$y'' + p(x)y' + q(x)y = r(x)$$

if $q(x) = 0$ we call it Homogeneous Part (1) of CH(3)

if $q(x) \neq 0$ we call it non Homogeneous Part (2) of CH(3)

The DE $(ay'' + by' + cy = 0)$ called with Constant coefficient

while $(y'' + p(x)y' + q(x)y = 0)$ called with variable coeff.

example 8. $y'' - 3y' + 4y = 0$ (Homo, 2nd, Linear with Constant coeff)

$2x^2y' - 3xy' + 7y = 0$ (Homo, 2nd, linear with variable coeff)

$y'' + y = x$ (non Homo, second, Linear)

An IVP for a Second Order ODE consist of

$$y'' = f(x, y, y')$$

$$(y(x_0) = \alpha, y'(x_0) = \beta) \text{ 2 ICs}$$

if $p(x), q(x), r(x)$ are cts on interval (I) containing (x_0) then the IVP has a Unique solution on (I) .

Example: Find the longest interval in which the solution of the IVP:

$$x(x-4)y'' + 3xy' + 4y = 2$$

$$y(3) = 0, y'(3) = 1 \quad \text{PS certain to exist.}$$

(1) Should be written on the general form:

$$y'' + \underbrace{\frac{3x}{x(x-4)}}_{p(x)} y' + \underbrace{\frac{4}{x(x-4)}}_{g(x)} y = \underbrace{\frac{2}{x(x-4)}}_{q(x)}$$

(2) make sure that $p(x), g(x), q(x)$ cts.

(check $p(x), g(x), q(x)$) $\Rightarrow (x=0, 4)$

$$\begin{array}{ccc} | & | & | \\ 0 & 3 & 4 \\ & \uparrow & \\ & x_0 & \end{array} \quad I = (0, 4) \quad \#$$

we will focus on DEs of the form:

$$(y'' + p(x)y' + g(x)y = q(x))$$

Superposition principle: if y_1, y_2 are solutions to the DE:

$$(y'' + p(x)y' + g(x)y = 0)$$

Then $y_1 \pm y_2$ is a solution & Cy_1 is a solution.

DE is linear homogeneous.

Example: Consider $y'' + y = 0$ ($y_1 = \sin(x), y_2 = \cos(x)$) are:

Solutions

So $y(x) = C_1 \sin(x) + C_2 \cos(x)$ is a solution.

note that $(C_1 y_1 + C_2 y_2)$ is called a linear combination of y_1, y_2 .

ضلع و جیبین مخالفین صل و (Sin x و X)

أذن نفس العائلة بوالله نفس Structure (اسكل) X و X^2

- if $f(x)$ and $g(x)$ are not linearly independent we say that they are linearly dependent ($f(x) = c \cdot g(x)$)

$$W[f, g] = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = (fg' - f'g)$$

Example: (1) W [Sin x, Cos x] =

Sin x	Cos x
Cos x	-Sin x

$$= -\sin x^2 - \cos x^2$$

$$= -(\sin^2 x + \cos^2 x) = -1 \neq 0 \text{ on } x \in (-\infty, \infty)$$

Thus, $(\sin x, \cos x)$ linearly independent on $(-\infty, \infty)$

$$(2) W[e^x, e^{-3x}] = \begin{vmatrix} e^x & e^{-3x} \\ e^x & -3e^{-3x} \end{vmatrix} = -3e^{-2x} - e^{-2x} = -4e^{-2x} \neq 0 \text{ linearly independent for any } x \in (-\infty, \infty)$$

• **Able's Theorem** Let y_1, y_2 be two solution to this DE: $(y'' + p(x)y' + q(x)y = 0)$ the $W[y_1, y_2] = C e^{-\int p(x) dx}$

• **Theorem** Let y_1, y_2 be two linearly independent solution to $(y'' + p(x)y' + q(x)y = 0)$, then $y(x) = C_1 y_1 + C_2 y_2$ is the general solution to this DE.

• $\{y_1, y_2\}$: form of a fundamental set of solution.

Example $y_1 = e^{2x}, y_2 = e^{-2x}$ are solutions to $(y'' - 4y = 0)$

$$W[y_1, y_2] = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -2 - 2 = -4 \neq 0$$

So y_1, y_2 are linearly independent on $(-\infty, \infty)$

• **general solution:**

$$y(x) = C_1 e^{2x} + C_2 e^{-2x}$$

• $\{e^{2x}, e^{-2x}\}$ a fundamental set of solution.

Example $(y'' - 9y = 0), y_1 = e^{3x}, y_2 = e^{-3x}$

$(W \neq 0)$ linearly independent solution to the DE.

$$\text{General solution} \Rightarrow y(x) = C_1 e^{3x} + C_2 e^{-3x}$$

مثال آخر: $(y'' - 9y = 0)$ مع الشروط الابتدائية $y(0) = 4, y'(0) = 5$

$$y(x) = 4 \sinh 3x - 5e^{-3x}$$

$$= 4 \frac{e^{3x} - e^{-3x}}{2} - 5e^{-3x} = 2e^{3x} - 7e^{-3x}$$

$C_1 \quad C_2$

Example 8 Consider the DE $(x^2 y'' - 3xy' + 4y = 0) (x > 0)$.

(1) Show that $y_1(x) = x^2$ and $y_2 = x^2 \ln x$ are solutions to the given DE.

(2) Show that y_1, y_2 are linearly independent.

(3) Write the general solution to the DE.

(4) Find the solution to the (IVP):

$$(x^2 y'' - 3xy' + 4y = 0)$$

$$y(1) = 2, \quad y'(1) = 3$$

$$(1) \quad y_1 = x^2 \Rightarrow y' = 2x, \quad y'' = 2 \quad (\text{م. أ. } x^2 \text{ حل (جزء) من (المعادلة)})$$

$$2x^2 - 6x^2 + 4x^2 = 0 \quad \checkmark \quad \text{then } x^2 \text{ is a solution.}$$

$$y_2 = x^2 \ln x \Rightarrow y' = 2x \ln x + x, \quad y'' = 2 + 2 \ln x + 1 = 2 \ln x + 3$$

$$2x^2 \ln x + 3x^2 - 6x^2 \ln x - 3x^2 + 4x^2 \ln x = 0 \quad \checkmark$$

then $x^2 \ln x$ is a solution.

$$(2) \quad W = \begin{vmatrix} x^2 & x^2 \ln x \\ 2x & x + 2x \ln x \end{vmatrix} = x^3 \neq 0 \quad \text{linearly independent on } (0, \infty)$$

$\{x^2, x^2 \ln x\}$ a fundamental set of solution.

(3) Thus general solution:

$$y(x) = C_1 x^2 + C_2 x^2 \ln(x). \quad \text{To the DE.}$$

(4) To find C_1, C_2 :

$$\text{Use } y(1) = 2 \Rightarrow C_1 + 0 = 2$$

$$\text{Use } y'(1) = 3 \Rightarrow C_2 = -1$$

The solution $\Rightarrow y(x) = 2x^2 - x^2 \ln(x)$. (particular solution to IVP)

Example 8 Consider the DE $(3x^2y'' + 11x'y' - 3y = 0)$, $(x > 0)$
 find a pair y linearly independent solution of the
 form $y(x) = x^r$. write the general solution to the
 DE. (نحتاج إلى زوج من الحلول المستقلة خطياً)
يكون أبغاي المعادلة

Solution (نبدأ بـ $y(x) = x^r$)
 $y'(x) = r x^{r-1}$
 $y''(x) = r(r-1) x^{r-2}$

- $y(x)$ is solution then it satisfies the DE.

$\Rightarrow (3x^2 r(r-1)x^{r-2} + 11x r x^{r-1} - 3x^r = 0)$
(نضع $x=0$ في المعادلة)
ببساطة $(3r(r-1)x^r + 11rx^r - 3x^r = 0)$

$x^r(3r^2 - 3r + 11r - 3 = 0)$

$x^r \neq 0$. This gives that $3\tilde{a}r^2 + 8\tilde{b}r - 3 = 0$

$\Delta = b^2 - 4ac = 64 - 4(3)(-3) = 100 > 0$

$r = \frac{-b \pm \sqrt{100}}{2a} = \frac{-4 \pm 5}{3}$

So $(r_1 = 1/3, r_2 = -3) \Rightarrow (x^{1/3}, x^{-3})$ are solution

$W[x^{1/3}, x^{-3}] = \begin{vmatrix} x^{1/3} & x^{-3} \\ \frac{1}{3}x^{-2/3} & -3x^{-4} \end{vmatrix} = -3x^{-11/3} - \frac{1}{3}x^{-1/3} \Rightarrow \underline{(-3 - 1/3)x^{-11/3}} \neq 0$

$y_1(x), y_2(x)$ are linearly independent solution. (نحتاج إلى زوج من الحلول المستقلة خطياً)

The general solution to the DE is $y(x) = C_1 y_1 + C_2 y_2$
 $= C_1 x^{1/3} + C_2 x^{-3}$

• Reduction of Order :

Let $y_1(x)$ be a known solution to the DE $(y'' + p(x)y' + g(x)y = 0)$

• We need a second linearly independent solution $y_2(x)$ to write the general solution.

• Reduction of Order method can help to find such $y_2(x)$.

Reduction of Order method assumes that $y_2(x) = \overset{\text{Function.}}{v(x)} y_1(x)$

To find $v(x)$, we use that $y_2(x) = v(x) y_1(x)$

$$y_2'(x) = v y_1' + v' y_1$$

$$y_2''(x) = v y_1'' + v' y_1' + v' y_1' + v'' y_1$$

So $(y_2' + p(x)y_2' + g(x)y_2 = 0)$

$$v y_1'' + 2 v' y_1' + v'' y_1 + p(x) (v y_1' + v' y_1) + g(x) (v y_1) = 0$$

$$v (y_1'' + p(x)y_1' + y_1 g(x)) + y_1 v'' + (2y_1' + p(x)y_1) v' = 0$$

$$y_1 v'' + (2y_1' + p(x)y_1) v' = 0$$

$$\int \frac{v''}{v} dx = \int \frac{(-2y_1' + p(x)y_1)}{y_1} dx \Rightarrow \ln v' = -(2 \ln y_1 + \int p(x) dx)$$

$$v' = e^{-2 \ln y_1 - \int p(x) dx} \Rightarrow \frac{2 \ln y_1}{e} \cdot \frac{-\int p(x) dx}{e}$$

$$\text{Now } \int v' = \int \frac{e^{-\int p(x) dx}}{(y_1)^2} dx \Rightarrow$$

$$v = \int \frac{e^{-\int p(x) dx}}{y_1^2} dx$$

(y_1 a given solution to $y'' + p(x)y' + g(x)y = 0$.)

• To find a second linearly independent solution we assume

$$y_2(x) = v(x) y_1(x) \Rightarrow \int \frac{e^{-\int p(x) dx}}{(y_1)^2} dx$$

Homogeneous Equations With Constant Coefficients:

The DE $(ay' + by' + Cy = 0)$ where a, b, C are constant is called "With Constant Coeff."

To solve such equations, we assume solutions of the form $(y = e^{rx})$

Now, $(y' = r e^{rx}, y'' = r^2 e^{rx})$ substituting in the Diff equ:

$$ar^2 e^{rx} + br e^{rx} + C e^{rx} = 0$$

$$e^{rx} (ar^2 + br + C) = 0$$

But $e^{rx} \neq 0$, then $ar^2 + br + C = 0$

this is a quadratic equ in (r) called the auxiliary equ.

To solve the aux equ (1) we first compute the discriminant

$$\Delta = b^2 - 4ac$$

- We have 3 cases: (1) $\Delta > 0$, (2) $\Delta = 0$, (3) $\Delta < 0$

Case (1): When $\Delta > 0$ we have distinct real roots $r = \frac{-b \pm \sqrt{\Delta}}{2a}$

$$\text{So } y_1 = e^{r_1 x}, y_2 = e^{r_2 x}$$

$$W(y_1, y_2) = \begin{vmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} \end{vmatrix} = r_2 e^{(r_1+r_2)x} - r_1 e^{(r_1+r_2)x} \neq 0$$

$\{e^{r_1 x}, e^{r_2 x}\}$ are set fundamental of solution.

the general solution: $y(x) = e^{r_1 x} C_1 + e^{r_2 x} C_2$

Case (2): When $\Delta = 0$ we have repeated roots $r_1 = r_2 = \frac{-b \pm \sqrt{\Delta}}{2a}$

$$\text{So } y_1(x) = e^{(-\frac{b}{2a})x}$$

reduction of order (if $r_1 = r_2$)

- We have a second linearly independent solution, Reduction of Order method is to be used to find y_2 .

$$y_2 = v(x) e^{(-b/2a)x} \quad (y'' + b y' + c y = 0)$$

$$v = \int \frac{e^{-\int \frac{b}{2a} dx}}{e^{(-b/2a)x}} dx = (x)$$

$$y_2 = x e^{-b/2a x} \quad (x) \text{ (بـ } x \text{)}$$

the general solution is $y(x) = C_1 e^{(-b/2a)x} + C_2 x e^{(-b/2a)x}$

Case (3) When $D < 0$, Here the roots of aux $r = \frac{-b \pm \sqrt{D}}{2a}$ (Complex)

Complex: $\sqrt{-1} = i$ or $i^2 = -1$

$$\alpha = \{ a + bi, a, b \in \mathbb{R} \}$$

Complex number.

So $r = \frac{-b \pm i\sqrt{D}}{2a}$ Written on the form $\alpha \pm \beta i$

$$= \left(\frac{-b}{2a} \right) \pm i \left(\frac{\sqrt{D}}{2a} \right)$$

$$y_1(x) = e^{(\alpha + \beta i)x}, \quad y_2 = e^{(\alpha - \beta i)x}$$

We will try to find real solution

recall Euler's formula $e^{ix} = \cos x + i \sin x$

$$\begin{aligned} e^{i\beta x} &= \cos \beta x + i \sin \beta x \\ e^{-i\beta x} &= \cos \beta x - i \sin \beta x \end{aligned} \quad \begin{aligned} e^{i\beta x} + e^{-i\beta x} &= 2 \cos \beta x \\ e^{i\beta x} - e^{-i\beta x} &= 2i \sin \beta x \end{aligned}$$

$$y_1 + y_2 = e^{(\alpha + \beta i)x} + e^{(\alpha - \beta i)x} = 2e^{\alpha x} \cos \beta x$$

$$y_1 - y_2 = e^{(\alpha + \beta i)x} - e^{(\alpha - \beta i)x} = 2i e^{\alpha x} \sin \beta x$$

So $\{ e^{\alpha x} \cos \beta x, e^{\alpha x} \sin \beta x \}$ are real solution to the DE.

the general solution $y(x) = C_1 e^{ax} \cos bx + C_2 e^{ax} \sin bx$

Summary

• $ay'' + by' + cy = 0$ we assume $y(x) = e^{rx}$

then r the aux eqn ($r^2 + br + c = 0$)

Solve the aux eqn for (r):

$\Delta > 0$, Two different roots $y_1 = e^{r_1 x}$, $y_2 = e^{r_2 x}$

$\Delta = 0$, repeated roots ($r_1 = r_2 = r$) $y_1 = e^{rx}$, $y_2 = x e^{rx}$

$\Delta < 0$, Complex roots ($r = \alpha \pm \beta i$) $y_1 = e^{\alpha x} \cos \beta x$, $y_2 = e^{\alpha x} \sin \beta x$

Example: Solve the following ODEs:

(1) $12y'' - 5y' - y = 0$

We seek solution of the form (e^{rx}) then r satisfies the aux eqn:

$$12r^2 - 5r - 1 = 0$$

$$\Delta = 25 - 4(12)(-1) = 121 > 0$$

$$r = \frac{5 \pm \sqrt{121}}{24}$$

($y_1 = e^{2/3x}$, $y_2 = e^{-1/4x}$) the general solution $\Rightarrow y(x) = C_1 e^{2/3x} + C_2 e^{-1/4x}$

(2) $y'' - 10y' + 25y = 0$

the aux eqn $\Rightarrow r^2 - 10r + 25 = 0$

$$\Delta = 100 - 4(1)(25) = 0 \text{ repeated roots } r_1 = r_2 = \frac{-b}{2a} = (5)$$

($y_1 = e^{5x}$, $y_2 = x e^{5x}$) the general $\Rightarrow y(x) = C_1 e^{(5x)} + C_2 x e^{(5x)}$

$$(3) 2y'' - 3y' - 4y = 0$$

$$\text{the aux equ} \Rightarrow 2r^2 - 3r - 4 = 0$$

$$\Delta = 9 - (4)(2)(4) = -23 < 0$$

$$\text{We have complex roots } r = \frac{3 \pm i\sqrt{23}}{4} = \left(\frac{3}{4}\right) \pm \left(\frac{\sqrt{23}}{4}\right)i$$

$$y_1 = e^{\frac{3}{4}x} \cos\left(\frac{\sqrt{23}}{4}x\right), \quad y_2 = e^{\frac{3}{4}x} \sin\left(\frac{\sqrt{23}}{4}x\right)$$

$$(4) y'' + 3y' = 0$$

$$\text{the aux equ} \Rightarrow r^2 + 3r = 0$$

$$r(r+3) = 0$$

$$(r=0, r=-3)$$

$$y_1 = e^{0x} = 1, \quad y_2 = e^{-3x} \quad \text{the general} \Rightarrow y(x) = C_1 + C_2 e^{-3x}$$

$$(5) y'' + 9y = 0$$

$$\text{the aux equ} \Rightarrow r^2 + 9 = 0$$

$$r = \pm \sqrt{-9} = \pm i3 \Rightarrow (0) \pm (3)i$$

$$y_1 = \cos(3x), \quad y_2 = \sin(3x)$$

$$\text{the general} \Rightarrow y(x) = \cos(3x) + \sin(3x)$$

Example 6 Find the Solution of the IVP $(y'' + 2y' + 4y = 0)$

$$(y(0) = 0, y'(0) = 2)$$

Solutions on the form e^{rx} where r satisfies the aux $\Rightarrow r^2 + 2r + 4 = 0$

$$\Delta = 4 - 4(2)(4)$$

$$\Delta = -12 < 0$$

$$r = \frac{-2 \pm i\sqrt{12}}{2} = -1 \pm i\sqrt{3}$$

(Complex)

$$y_1 = e^{-x} \cos(\sqrt{3}x), \quad y_2 = e^{-x} \sin(\sqrt{3}x)$$

$$\text{the general} \Rightarrow y(x) = C_1 e^{-x} \cos(\sqrt{3}x) + C_2 e^{-x} \sin(\sqrt{3}x)$$

the I.C. ($y(0)=0$) $\Rightarrow e^0 (\cos(0) C_1 + 0) = 0$

$C_1 = 0$

$\therefore Y(x)$ became $= e^{-x} \sin(\sqrt{3}x) C_2$.

Now, $y'(x) = C_2 \sqrt{3} e^{-x} \cos(\sqrt{3}x) - C_2 e^{-x} \sin(\sqrt{3}x)$

($y'(0) = 2$) $\Rightarrow 2 = \sqrt{3} C_2 - 0$

$C_2 = \frac{2}{\sqrt{3}}$

Example 8 Solve the IVP ($2y'' + y' - y = 0$) ($y(1) = 1$, $y'(1) = -2$)

the aux equ $= 2r^2 + r - 1 = 0$

$D = 1 - 4(2)(-1) = 9 > 0$

$r = \frac{-1 \pm 3}{4}$ ($y_1 = e^{1/2x}$, $y_2 = e^{-x}$)

the general solution $y(x) = e^{1/2x} C_1 + e^{-x} C_2$

($y(1) = 1$) $\Rightarrow C_1 = -2/3 e^{1/2}$

($y'(1) = -2$) $\Rightarrow C_2 = 5/3 e$

Cauchy - Euler equations

A 2nd Order DE for the form
 $ax^2y'' + bxy' + Cy = 0, x > 0$ is called Cauchy Euler equation.

To solve it, we assume solutions in the form $y(x) = x^r$.

$$y(x) = x^r$$

$$y'(x) = r x^{r-1}$$

$$y''(x) = r(r-1)x^{r-2}$$

now substituting on the DE: $ax^2 r(r-1)x^{r-2} + bx r x^{r-1} + Cx^r = 0$

$$ar(r-1)x^r + brx^r + Cx^r = 0$$

$$x^r [ar^2 - ar + br + C] = 0$$

$$x^r [ar^2 + (b-a)r + C] = 0$$

Since $x > 0$, then $x^r \neq 0$

Thus, $[ar^2 + (b-a)r + C = 0]$ This quadratic equation is called the auxiliary equation.

• Solve the aux:

Case (1): Two real distinct roots r_1, r_2 .

$$y_1 = x^{r_1}, y_2 = x^{r_2}$$

$$W[x^{r_1}, x^{r_2}] = \begin{vmatrix} x^{r_1} & x^{r_2} \\ r_1 x^{r_1-1} & r_2 x^{r_2-1} \end{vmatrix} = r_2 x^{r_1+r_2-1} - r_1 x^{r_2+r_1-1} = (r_2 - r_1) x^{r_1+r_2-1} \neq 0 \text{ for } x > 0$$

The general solution is $y(x) = C_1 x^{r_1} + C_2 x^{r_2}$

• Case (2): Repeated roots

$$r_1 = r_2 = -\frac{(b-a)}{2a}$$

$$(y_1(x) = x^{-(\frac{b-a}{2a})})$$

* reduction of order method is to be used to find a 2nd linearly independent solution $y_2(x)$.

$$y_2(x) = V(x) x^{-(\frac{b-a}{2a})}$$

$$(ax^2y' + by' + cy = 0)$$

$$(y'' + \underbrace{\frac{b}{ax}}_{P(x)} y' + \frac{c}{ax^2} = 0)$$

$$V(x) = \int \frac{e^{-\int P(x) dx}}{y_1^2} dx$$

$$= \int \frac{e^{-\int \frac{b}{ax} dx}}{\frac{x^{-(b-a)}{x^a}}{x^a}} dx = \int \frac{x^{-b/a}}{x^{-b/a} \cdot x} dx = -\ln(x)$$

So $y_2(x) = \ln(x) y_1$, The general solution is $y(x) = C_1 x^{-(\frac{b-a}{2a})} + C_2 x^{-(\frac{b-a}{2a})} \ln x$

• Case (3): Complex roots

$$r = \alpha \pm \beta i$$

$$y_1(x) = x^{\alpha + \beta i}$$

$$y_2(x) = x^{\alpha - \beta i}$$

are Complex Solution.

• NOW we seek real solution:

Euler formula $e^{iz} = \cos z + i \sin z$

$$e^{iBx} = \cos Bx + i \sin Bx$$

$$e^{-iBx} = \cos Bx - i \sin Bx$$

$$e^{iBx} + e^{-iBx} = 2 \cos Bx$$

$$e^{iBx} - e^{-iBx} = 2i \sin Bx$$

$$y_1(x) + y_2(x) = x^{\alpha} x^{\beta i} + x^{\alpha} x^{-\beta i} \Rightarrow x^{\alpha} (x^{\beta i} + x^{-\beta i})$$

$$= e^{\alpha \ln x} (e^{iB \ln x} + e^{-iB \ln x}) = 2 e^{\alpha \ln x} \cos(B \ln x)$$

$$y_1(x) - y_2(x) = \rho i e^{\alpha \ln x} \sin(\beta \ln x)$$

$$\left. \begin{aligned} \frac{1}{2}(y_1 + y_2) &= x^{\alpha} \cos(\beta \ln x) \\ \frac{1}{2}i(y_1 - y_2) &= x^{\alpha} \sin(\beta \ln x) \end{aligned} \right\} \text{Real Solutions}$$

• To solve $(ax^2y'' + bxy' + Cy = 0, x > 0)$

Solution in the form $y(x) = x^r$, r satisfies the aux equ:

$$(ar^2 + (b-a)r + C = 0)$$

$$\text{if } (r_1, r_2) \quad y(x) = C_1 x^{r_1} + C_2 x^{r_2}$$

$$(r_1 = r_2) \quad y(x) = C_1 x^{r_1} + C_2 x^{r_1} \ln x$$

$$(r_1, r_2) \quad y(x) = C_1 x^{\alpha} \cos(\beta \ln x) + C_2 x^{\alpha} \sin(\beta \ln x)$$

Example 8 Solve the DE $(2x^2y'' + 3xy' - y = 0)$

This is Euler equ we assume $y(x) = x^r$ where r satisfies the aux equ $(2r^2 + r - 1 = 0)$

$$D = b^2 - 4ac = 9 > 0 \quad \text{Two real solutions, different.}$$

$$r_1 = \frac{1 + \sqrt{9}}{4}, \quad r_2 = \frac{1 - \sqrt{9}}{4}$$

$$r_1 = \frac{1}{2}, \quad r_2 = -1$$

$$y_1(x) = x^{-1}, \quad y_2(x) = x^{\frac{1}{2}}$$

$$\text{The general solution} \Rightarrow y(x) = C_1 x^{\frac{1}{2}} + C_2 x^{-1}$$

example 8 Solve the Cauchy - Euler equ ?

$$x^2 y'' - 4xy' + 4y = 0$$

$$a=1, B=-4, C=4$$

the aux 8 $r^2 - 4r + 4 = 0$

$$D=0 \quad r_1 = r_2 = 2 \text{ repeated roots}$$

$$y_1(x) = x^2, \quad y_2(x) = x^2 \ln(x)$$

the general solution 8 $y(x) = C_1 x^2 + C_2 x^2 \ln x$

Non-Homogeneous Equations :

it has the form : $y'' + p(x)y' + q(x)y = g(x)$.

where $g(x) \neq 0$

عشائر تفصل (non homo) ، المعادلة ، homo ، متجانسة
للمعادلة ، (non homo) - (مجانسة)

The corresponding homogeneous equ results when $g(x) = 0$:

$$y'' + p(x)y' + q(x)y = 0$$

has the solution :

$$y_h(x) = C_1 y_1 + C_2 y_2$$

homogeneous

هذا المعادلة المتجانسة ليس بها حد ثابت
الحل (y_h)

Any solution to the non homogeneous equ is called a particular solution $\rightarrow (y_p(x))$ - هذا الحل للمعادلة non

التي هي ، البسط الأول (بالتي) ، مع غير كافي الكتابة ، general solution

Theorem : If $y_1(x), y_2(x)$ are solutions to $y'' + p(x)y' + q(x)y = g(x)$ then $y_1(x) - y_2(x)$ is a solution to $y'' + p(x)y' + q(x)y = 0$

إذا عندك حلين للمعادلة non homo فإن حاصل طرحهما حل للمعادلة homo المتجانسة ، لأن ناتج تعويض $y_1(x)$ في المعادلة non homo سيكون $g(x)$ (طرف المعادلة) وكذلك عند تعويض $y_2(x)$ وعند طرح g منه سينتج 0 (طرف المعادلة) (homo) :
يعني $(y_1(x) - y_2(x))$ لا تساوي $2g(x)$ وهذا لا يفيدنا

Theorem : If y_1, y_2 are solutions to $y'' + p(x)y' + q(x)y = 0$ and $y_p(x)$ is a particular soln to $y'' + p(x)y' + q(x)y = g(x)$

Then $y(x) = C_1 y_1 + C_2 y_2 + y_p(x)$ is the general solution to $y'' + p(x)y' + q(x)y = g(x)$

Proof : Let $\phi(x)$ be any solution to $y'' + p(x)y' + q(x)y = g(x)$
 (بما أنه حل للمعادلة غير المتجانسة $y'' + p(x)y' + q(x)y = g(x)$)

• $\phi(x), y_p(x)$ are solutions to the corresponding homo-equ.

$$\phi(x) - y_p(x) = C_1 y_1 + C_2 y_2$$

$$\phi(x) = C_1 y_1 + C_2 y_2 + y_p(x)$$

Summary :

To solve $y'' + p(x)y' + q(x)y = g(x)$.

Step 1: Solve the corresponding homogenous equ :

$$y'' + p(x)y' + q(x)y = 0 \Rightarrow y_h(x) = C_1 y_1 + C_2 y_2$$

(With constant or Cauchy) ↗

Step 2: Try to find a particular solution $y_p(x)$ to the non homo-equ.

Step 3: The general solution to the non-homo equ :

$$y(x) = y_h(x) + y_p(x)$$

• To find a particular solution, we use one of the methods

1) Method of Undetermined Coefficient.

2) Method of variation of parameters.

Method of Undetermined Coefficients

This method is used to find a particular solution to the non-homogeneous equation $ay'' + by' + cy = g(x)$ (حالتنا مع Const. Coeff)

The method works when $g(x)$ is exponential (e^{ax}) or polynomial ($a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$) or Sine or Cosine or addition or multiplication of such function.

example 8 $g(x) = 10e^{3x}$ ✓, $g(x) = \sin(3x) - \cos(2x)$ ✓, $g(x) = \frac{1}{x+2}$
 we can use it. we can use it $\sin(bx), \cos(bx)$ X can't
 $g(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$ X we can't use it.

المسألة بين كثير الحدود والـ $\sin, \cos$ نوعه فقط الممنوع، ليس يمكنه لأي الاقترانات، ولا مشتقاته.
 يفضل انضها أو كنهه نفس الحالة

$$g(x) = \sec(x) = \frac{1}{\cos(x)} \quad X, \quad g(x) = e^{3x^2} \quad X$$

$$g(x) = 1/e^x = e^{-x} \quad a = -1 \quad \checkmark, \quad g(x) = x^2 e^{-3x} \sin(4x) \quad \checkmark$$

$$g(x) = e^{3x} + 10x + \sin(3x) \quad \checkmark \quad \text{مسمح جميعهم}$$

عني قيان (هنا الطريقة) لان في (1) المعطيات تكون ثوابت (2) $g(x)$ تكون على احدى (لعمري) المذكورة.

$g(x)$	$y_p(x)$
(1) Ae^{ax} ex & Ze^x	Be^{ax} ex & Be^x (الاسم نفسه لاننا لم نكتبه) (2) $P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ (3) $C \sin(bx)$ or $C \cos(bx)$ ex & $\sin 3x$
(4) $e^{ax} P_n(x)$	$e^{ax} (A_n x^n + A_{n-1} x^{n-1} + \dots + A_1 x + A_0)$ (2) $A_1 \cos(bx) + A_2 \sin(bx)$ ex & $A \cos(3x) + A \sin(3x)$

فالجواب الثاني
 لاننا لم نكتبه في التوسيع
 في المثال الثاني
 في المثال الثاني

$$(7) e^{ax} p_n(x) \sin(bx) \text{ or } e^{ax} p_n(x) \cos(bx) \quad e^{ax} [(A_n x^n + A_{n-1} x^{n-1} + \dots + A_1 x + A_0) \sin(bx) + (B_n x^n + B_{n-1} x^{n-1} + \dots + B_1 x + B_0) \cos(bx)]$$

٢٧٠ نفس بجمع .

$$ay'' + by' + cy = g(x)$$

Corresponding

homo-equ

$$ay'' + by' + cy = 0$$

$$y_h = C_1 y_1 + C_2 y_2$$

$y_p(x)$

General Solution

$$Y(x) = y_h(x) + y_p(x)$$

- Determine the form of a particular solution of the given DE. Don't evaluate the coefficients.

$$(1) y'' - 2y' - 3y = 3e^{2x}$$

the corresponding homo-equ ($y'' - 2y' - 3y = 0$)

the aux equ: $r^2 - 2r - 3 = 0$

$$(r-3)(r+1) = 0$$

$$r_1 = 3, r_2 = -1$$

$$y_1(x) = e^{3x}, y_2(x) = e^{-x}$$

Here $g(x)$ is $= 3e^{2x}$

So the first guess of $y_p(x)$ is $y_p(x) = Ae^{2x}$

e^{2x} isn't solution to y_1 or y_2 then $y_p(x) = Ae^{2x}$.

$$(2) y'' - y' = 3x^2 - 5$$

the corresponding homo-equ ($y'' - y' = 0$)

has aux $r^2 - r = 0$

$$r_1 = 0, r_2 = 1$$

$$y_1(x) = e^x, y_2(x) = 1$$

Here $g(x) = 3x^2 - 5$

So the first guess of $y_p(x)$ is $y_p(x) = A_2 x^2 + A_1 x + A_0$.

But the 3rd term is similar to $y_2(x)$ so $y_p(x) = A_2 x^3 + A_1 x^2 + A_0 x$

$$(3) Y'' + 4Y = 7 \sin(2x)$$

the corresponding homoequ $y'' + 4y = 0$

$$\text{aux } r^2 + 4 = 0$$

$$r = 0 \pm 2i$$

$$\alpha = 0, \beta = 2$$

$$Y_1(x) = e^{\alpha x} \cos(\beta x) = \cos(2x)$$

$$Y_2(x) = \sin(2x)$$

$$\text{Here } g(x) = 7 \sin 2x$$

An initial guess of $y_p(x)$ is $y_p(x) = A \cos(2x) + B \sin(2x)$

But $\cos(2x)$ similar to y_1 , so $y_p(x) = x(A \cos(2x) + B \sin(2x))$

$$(4) Y'' - 3Y' - 4Y = 3e^{-x} + 2\sin x - 8e^x \cos(2x)$$

the corresponding hom-equ $y'' - 3y' - 4 = 0$

$$\text{aux } r^2 - 3r - 4 = 0$$

$$r_1 = 4, r_2 = -1$$

$$y_1(x) = e^{4x}, y_2(x) = e^{-x}$$

$$\text{Here } g(x) = \underbrace{3e^{-x}}_{g_1} + \underbrace{2\sin x}_{g_2} - \underbrace{8e^x \cos(2x)}_{g_3}$$

$$g_1(x) = 3e^{-x}$$

$$y_{p1}(x) = A_1 x e^{-x}$$

$$g_2(x) = 2\sin x$$

$$y_{p2}(x) = A_2 \cos x + A_3 \sin x$$

$$g_3(x) = -8e^x \cos(2x)$$

$$y_{p3}(x) = e^x (A_4 \cos 2x + A_5 \sin 2x)$$

$$\text{So } y_p(x) = y_{p1} + y_{p2} + y_{p3}$$

$$(5) y'' + 2y' + 2y = 9e^{-x}(x^2 - 3)\sin x$$

The corresponding homo-equ is $y'' + 2y' + 2y = 0$

$$\text{has aux} \Rightarrow r^2 + 2r + 2 = 0$$

$$\Delta = -4 < 0$$

$$r = \frac{-2 \pm \sqrt{-4}}{2(1)} = -1 \pm i$$

$$y_1(x) = e^{-x} \cos x, \quad y_2 = e^{-x} \sin x$$

Here $g(x) = 9e^{-x}(x^2 - 3)\sin x$, An initial guess to the form of $y_p(x)$ is

$$y_p(x) = x e^{-x} [(A_2 x^2 + A_1 x + A_0) \cos x + (B_2 x^2 + B_1 x + B_0) \sin x]$$

example 8 Find the general solution of the non-hom equ :

$$y'' - 2y' + y = 3e^x$$

associate the corresponding homo-equ is : $y'' - 2y' + y = 0$

We seek solution in the form $y(x) = e^{rx}$ then r satisfying the aux equ

$$r^2 - 2r + 1 = 0$$

$$(r-1)(r-1) = 0 \quad \text{repeated roots} \quad y_1(x) = e^x, \quad y_2(x) = x e^x$$

Thus $y_h(x) = C_1 e^x + C_2 x e^x$

To find particular solution, $y_p(x)$ we will use the method of Undetermined Coefficient Here, $g(x) = 3e^x$

Our initial guess is

$$y_p(x) = A x^2 e^x$$

To find A :

$$y_p'(x) = A(x^2 e^x + 2x e^x) = A e^x (x^2 + 2x)$$

$$y_p''(x) = A[e^x(2x+2) + e^x(x^2+2x)] = A e^x (x^2 + 4x + 2)$$

Now, $y_p(x)$ satisfies $y_p'' - 2y_p' + y_p = 3e^x$

$$A e^x (x^2 + 4x + 2) - 2 A e^x (x^2 + 2x) + A x^2 e^x = 3e^x$$

$$2 A e^x = 3e^x \Rightarrow A = 3/2$$

So $y_p(x) = 3/2 x^2 e^x$

The general solution to the non-homo DE is

$$y(x) = y_h(x) + y_p(x) = C_1 e^x + C_2 x e^x + 3/2 x^2 e^x$$

example 8 find the general solution to the DE :

$$y'' + 2y' + y = x^2 + 1 - e^x$$

The corresponding homo equ is $y'' + 2y' + y = 0$

the aux $r^2 + 2r + 1 = 0$

$(r+1)(r+1) = 0$ repeated roots $y_1(x) = e^{-x}$, $y_2(x) = x e^{-x}$

$$y_h(x) = C_1 e^{-x} + C_2 x e^{-x}$$

Here $g(x) = \underbrace{x^2 + 1}_{g_1} - \underbrace{e^x}_{g_2} = g_1(x) + g_2(x)$

$y_{p1}(x) = Ax^2 + Bx + C$, $y_{p2}(x) = D e^x$

So $y_p(x) = Ax^2 + Bx + C + D e^x$

<Next we will determine A, B, C, D>

$y'_p(x) = 2Ax + B + D e^x$

$y''_p(x) = 2A + D e^x$

y_p satisfies $y''_p + 2y'_p + y_p = x^2 + 1 - e^x$

$$2A + D e^x + 4Ax + 2B + 2D e^x + Ax^2 + Bx + C + D e^x = x^2 + 1 - e^x$$

became : $Ax^2 + (4A+B)x + (2A+2B+C) + 4D e^x = x^2 + 1 - e^x$

Comparing Coeffs:

x^2 : $A = 1$

x : $4A + B = 0 \Rightarrow B = -4$

x^0 : $2A + 2B + C = 1 \Rightarrow C = 7$

e^x : $4D = -1 \Rightarrow D = -1/4$

Now $y_p(x) = x^2 - 4x + 7 - 1/4 e^x$

the general solution to the non-homo is $y(x) = y_h + y_p$

$$y(x) = C_1 e^{-x} + C_2 x e^{-x} + x^2 - 4x + 7 - 1/4 e^x$$

Contg $y(0)=0, y'(0)=2$

$$C_1 + 7 - 1/4 = 0$$

$$C_1 = 1/4 - 7.$$

• Use the method of Undetermined Coefficient to find the form of a particular Solution.

$$x^2 y'' - 3xy' - 5y = 2x^5 \ln x + x^{-1}$$

Every Cauch - Euler equ $ax^2 y'' + bxy' + cy = 0, x > 0$

Can be converted to an equ with constant coefficient let $x = e^t$ or $t = \ln x$.

$$y'' = \frac{d^2 y}{dx^2}, y' = \frac{dy}{dx}$$

$$y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{x}$$

$$y'' = \frac{d}{dx} \left(\frac{1}{x} \frac{dy}{dt} \right)$$

$$= \frac{1}{x} \left(\frac{d^2 y}{dt^2} \cdot \frac{1}{x} \right) - \frac{1}{x^2} \frac{dy}{dt}$$

$$= \frac{1}{x^2} \frac{d^2 y}{dt^2} - \frac{1}{x^2} \frac{dy}{dt}$$

Now: $ax^2 y'' + bxy' + cy = 0$

$$ax^2 \left(\frac{1}{x^2} y'' - \frac{1}{x^2} y' \right) + bx \frac{1}{x} y' + cy = 0$$

$$ay'' - ay' + by' + cy = 0$$

$$ay'' + (b-a)y' + cy = 0$$

$$x^2 y'' - 3xy' - 5y = 2x^5 \ln x + x^{-1}$$

$$a=1, b=-3, c=-5$$

⇒

the Cauchy-Euler eqn became after letting $x = e^t$ or $t = \ln x$

$$y'' - 4y' - 5y = 2te^{5t} + e^{-t}$$

$$y'' - 4y' - 5y = 0$$

$$\text{aux } r^2 - 4r - 5 = 0 \quad (\text{different roots})$$

$$r_1 = 5, r_2 = -1$$

$$y_1 = e^{5t}, y_2 = e^{-t}$$

$$y_p(t) = te^{5t}(At+B) + Cte^{-t}$$

$$y_p(x) = \ln x \cdot x^5 (A \ln x + B) + C \ln x \cdot x^{-1}$$

• Variation of parameter &

• to solve the non homo DE &

$$y'' + p(x)y' + q(x)y = g(x)$$

• the need to solve the corresponding eqn:

$$y_h(x) = C_1 y_1 + C_2 y_2$$

• then we need to find a particular solution $y_p(x)$ then the general solution is $y(x) = y_h(x) + y_p(x)$

• method of Undetermined Coeff has restrictions

$$ay'' + by' + cy = g(x) \quad \text{where } g(x) = \sin ax, e^{ax}, p_n(x), \cos ax$$

• we need more general method to find $y_p(x)$ for any eqn of the form $y'' + p(x)y' + q(x)y = g(x)$

• This method is called variation of parameters, the method assume $y_p(x) = u_1(x)y_1 + u_2(x)y_2$, (y_1, y_2) are the solutions to the corresponding eqn

$$\text{Now, } y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

$$y_p'(x) = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$$

$$\text{(assume } u_1'y_1 + u_2'y_2 = 0) \Rightarrow y_p'(x) = u_1y_1' + u_2y_2'$$

$$y'' = u_1y_1'' + u_1'y_1' + u_2y_2'' + u_2'y_2'$$

$$\text{Sub in } y'' + p(x)y' + q(x)y = g(x)$$

$$C_1 u_1 [y_1'' + p(x)y_1' + q(x)y_1] + C_2 [y_2'' + p(x)y_2' + q(x)y_2] + u_1'y_1' + u_2'y_2' = g(x)$$

$$C_1 u_1'y_1' + C_2 u_2'y_2' = g(x)$$

$$(y_1, y_2 \text{ is } 1, 1)$$

• Solve the system of u_1', u_2' &

$$u_1'(x) = \frac{-y_2 g(x)}{W(y_1, y_2)} \quad u_2'(x) = \frac{y_1 g(x)}{W(y_1, y_2)}$$

$$U_1 = - \int \frac{y_2 g(x)}{W(y_1, y_2)} dx, \quad U_2 = \int \frac{y_1 g(x)}{W(y_1, y_2)} dx$$

$$Y_p(x) = U_1(x) y_1(x) + U_2(x) y_2(x)$$

Ex 8 find the general solution $y'' + y = \tan x$, $x \in (-\pi/2, \pi/2)$

the corresponding homogene $y'' + y = 0$

has aux eq $r^2 + 1 = 0$

$$r = \pm i\sqrt{1}$$

$$0 \pm i$$

$$y_1 = \cos(x), \quad y_2 = \sin(x)$$

$$\text{So } y_h(x) = C_1 \cos x + C_2 \sin(x)$$

Here $g(x) = \tan x$

$$W[y_1, y_2] = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

method of variation of parameter assumes $U_1 = \cos x$, $U_2 = \sin x$

$$U_1 = - \int \frac{\sin(x) \tan x}{1} dx = - \ln |\sec x - \tan x| - \sin x$$

$$U_2 = \int \cos(x) \tan x dx = - \cos x$$

$$(- \cos x \ln |\sec x + \tan x|)$$

$$\text{So } y_p(x) = - \cos x \ln |\sec x + \tan x| + \cos x \sin x - \cos x \sin x$$

The general solus $Y_h(x) + Y_p(x) = C_1 \cos x + C_2 \sin x - \cos x \ln |\sec x + \tan x|$

Example 2 Find the general solution to $(x^2y'' - 4xy' + 6y = x^3 + 1)$ x70

the corresponding homoequ: $x^2y'' - 4xy' + 6y = 0$

Cauchy - Euler $a=1, b=-4, c=6$

Solution are in the $y(x) = x^r$

r satisfies $ar^2 + (b-a)r + c = 0$

$$r^2 + -5r + 6 = 0$$

$$(r-3)(r-2) = 0 \Rightarrow r = 2, 3$$

$$y_1(x) = x^2, y_2(x) = x^3$$

$$\text{So } y(x) = C_1 x^2 + x^3 C_2$$

the DE standard form $y'' - \frac{4}{x}y' + \frac{6}{x^2}y = \frac{x^3+1}{x^2}$

$$g(x) = \frac{x^3+1}{x^2}$$

$$W[y_1, y_2] = \begin{vmatrix} x^3 & x^2 \\ 3x^2 & 2x \end{vmatrix} = 2x^4 - 3x^4 = -x^4$$

Variation of parameter Assume:

$$y(x) = u_1(x)x^3 + u_2(x)x^2$$

$$u_1(x) = - \int \frac{x^2 \frac{x^3+1}{x^2}}{-x^4} = \ln x - \frac{x^3}{3}$$

$$\int \frac{x^3 \frac{x^3+1}{x^2}}{-x^4} = -x + \frac{1}{2}x^{-2}$$

$$y_p(x) = x^3 \ln x - \frac{1}{3}x^3 - x + \frac{1}{2}x^{-2} = x^3 \ln x - x^3 + \frac{1}{6}$$

the general solution $C_1 x^3 + C_2 x^2 + x^3 \ln x - x^3 + \frac{1}{6}$

$$x^2y'' - 4xy' + 6y = x^3 + 1$$

$$x = e^t, t = \ln x$$

$$y'' - 5y' + 6y = e^{3t} + 1 \quad \text{Undetermined coeff}$$

$$e^{3t}, e^{0t} \Rightarrow n = 3, 0$$

CH 4 Higher Order ODEs.

the DE $y^{(n)} = F(x, y, y', \dots, y^{(n-1)})$ is the general form of an ODE of Order (n) , if $(n) \geq 2$ the DE is called "a higher Order DE".

$$y^{(n)} + P_{n-1}(x)y^{(n-1)} + P_{n-2}(x)y^{(n-2)} + \dots + P_1(x)y' + P_0(x)y = g(x)$$

Linear DE of Order (n) .

if $g(x) = 0 \Rightarrow$ we call it homogeneous DE.

if $g(x) \neq 0 \Rightarrow$ we call it non-homogeneous DE.

- if coefficients are constants

$$ay^{(n)} + by^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

we call it with constant coefficients otherwise we call it with variable coeffs.

For example: (1) $y^{(3)} - 4y'' + 7y = 0 \Rightarrow$ "linear, 3 order, constant homo"

$$(2) x^4 y^{(4)} + 3x^3 y^{(3)} + 2x^2 y^{(2)} + x y' + 10y = 0$$

"linear, Cauchy Euler, 4 Order, Homo"

$$y^{(n)} + \frac{P_{n-1}(x)}{x} y^{(n-1)} + \frac{P_{n-2}(x)}{x^2} y^{(n-2)} + \dots + \frac{P_1(x)}{x^{n-1}} y' + \frac{P_0(x)}{x^n} y = \frac{g(x)}{x^n}$$

with $y(x_0) = \alpha_0, y'(x_0) = \alpha_1, \dots, y^{(n-1)}(x_0) = \alpha_{n-1}$

if $P_{n-1}(x), P_{n-2}(x), \dots, P_1(x), P_0(x), g(x)$ are cts on an open interval containing x_0 then the DE has a unique solution on I .

Example: Find the largest interval on which the IVP

$$y^{(3)} - \ln x y'' + \frac{1}{x-1} y' + 10y = \sin(x)$$

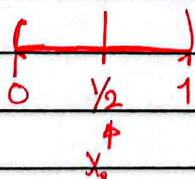
$y(1/2) = 0, y'(1/2) = 2, y''(1/2) = -1$ is guaranteed to have a unique solution.

$$\ln x \Rightarrow x > 0$$

$$\frac{1}{x-1} \Rightarrow x \neq 1$$

$$10 \Rightarrow x \in \mathbb{R}$$

$$\sin x \Rightarrow x \in \mathbb{R}$$



So, I is $(0, 1)$

Def: the functions f_1, f_2, f_3 are set to be "linearly dependent" on (I) if there exist constant C_1, C_2, C_3, C_n all not zero such that " $C_1 f_1 + C_2 f_2 + C_3 f_3 + \dots + C_n f_n = 0$ "

if they are not dependent we say that they are "linearly independent"

$f_1 = C_2 f_2 + C_3 f_3 + \dots + C_n f_n$ $\Leftarrow$ Linear combination of f_3, f_2 & $\dots$ f_n $\Leftarrow$ f_1 is dependent on $f_2, f_3, \dots, f_n$

if $y_1, y_2, \dots, y_n$ are linearly independent solution to

$$y^{(n)} + P_{n-1}(x)y^{(n-1)} + P_{n-2}(x)y^{(n-2)} + \dots + P_1(x)y' + P_0(x)y = g(x)$$

then $y_1, y_2, \dots, y_n$ form of fundamental set of solutions this means that the general solution is

$$y(x) = C_1 y_1 + C_2 y_2 + C_3 y_3 + \dots + C_n y_n$$

Homogeneous eqn with Constant Coefficients

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

we seek solution of the form e^{rx} .

$$y' = r e^{rx}$$

$$y'' = r(r-1) e^{rx}$$

$$y''' = ?$$

$$\text{Sub. at the DE} \Rightarrow \text{become } a_n r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0 = 0$$

$$e^{rx} [a_n r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0] = 0$$

but $e^{rx} \neq 0$

$$\text{then } [a_n r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0] = 0 \text{ called:}$$

"the aux eqn to the DE"

Example (1) Find the general solution to the DE :

" $y'' + 3y' - 10y = 0$ " $\Rightarrow$ 3rd order homo linear with const. co

We assume sol. in the form $y(x) = e^{rx}$

then r satisfies the aux equ $r^3 - 3r^2 - 10r = 0$

$$r(r^2 - 3r - 10) = 0$$

" $r = 0, r = -5, r = 2$ "

we have three distinct real roots

$y_1(x) = e^{0x} = 1, y_2(x) = e^{-5x}, y_3(x) = e^{2x}$

the general solution : $y(x) = C_1 e^{-5x} + C_2 + C_3 e^{2x}$

Example (2) " $y^{(4)} + 2y'' + y = 0$ " $\Rightarrow$ 4th order, homo, linear with const. coeff.

the aux equ $\Rightarrow r^4 + 2r^2 + 1 = 0$

$$(r^2 + 1)^2 = 0$$

repeated complex roots $r = \pm i, \alpha = 0, \beta = 1$

$y_1(x) = e^{0x} \cos x, y_2(x) = e^{0x} \sin x$

$y_3(x) = x \cos x, y_4(x) = x \sin x$

الحل العام (x) = ...

the general solution : $y(x) = \cos x C_1 + \sin x C_2 + x \cos x C_3 + x \sin x C_4$

Example (3) $9y^{(5)} - 6y^{(4)} + y^{(3)} = 0$

the aux equ : $9r^5 - 6r^4 + r^3 = 0$

$$r^3(9r^2 - 6r + 1) = 0$$

$r^3 = 0$ $\Rightarrow$ triple root at $r = 0$

$(3r - 1)^2 = 0 \Rightarrow r = 1/3$ (double root)

$(r_1 = r_2 = r_3 = 0, r_4 = r_5 = 1/3)$

$y_1(x) = 1, y_2(x) = x, y_3(x) = x^2, y_4 = 1/3, y_5 = 1/3 x$

the general solution : $y(x) = C_1 + x C_2 + x^2 C_3 + 1/3 C_4 + 1/3 x C_5$

Example (4) : $y^{(3)} + y' - 10y = 0$

the aux equ: $r^3 + r - 10 = 0$

$$\begin{matrix} \wedge \\ \pm 1 \pm 10 \pm 2 \end{matrix}$$

(2) $r = -10$ (بجرب عوامل 10 والـ 1)

So By fail and trail, we found that $r=2$ is a root. So $(r-2)$ is a factor of $(r^3 + r - 10 = 0)$

$$r^3 + r - 10 = (r-2) (\quad)$$

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$$\begin{array}{r} r^2 + 2r + 5 \\ r-2 \overline{) r^3 - r - 10} \end{array}$$

$$(r^2 + 2r + 5) = 0$$

$$\begin{array}{r} -r^3 - 2r^2 \\ \hline 2r^2 + r - 10 \\ -2r^2 - 4r \\ \hline 5r - 10 \\ -5r - 10 \\ \hline 0 \end{array}$$

$$r = \frac{-2 \pm \sqrt{-16}}{2} = -1 \pm 2i$$

$$r_1 = 2, r_{2,3} = -1 \pm 2i$$

$$\begin{aligned} y_1(x) &= e^{2x} \\ y_2(x) &= e^{-x} \cos 2x \\ y_3(x) &= e^{-x} \sin 2x \end{aligned}$$

the general solution: $y(x) = e^{2x} C_1 + e^{-x} \cos 2x C_2 + e^{-x} \sin 2x C_3$

Example (5) : $y^{(5)} + 3y^{(4)} - 2y^{(3)} - 4y'' = 0$

the aux equ: $r^5 + 3r^4 - 2r^3 - 4r^2 = 0$

$$r^2 (r^3 + 3r^2 - 2r - 4) = 0$$

$r_1 = r_2 = 0$, By fail and trail $r = -1$ is a root. $(r+1)$ is a factor

(بجرب عوامل 10 والـ 1)

$$r^3 + 3r^2 - 2r - 4 = (r+1)(r^2 + 2r - 4)$$

$$r = \frac{-2 \pm \sqrt{20}}{2} = -1 \pm \sqrt{5}$$

$$r_4 = -1 + \sqrt{5}, r_5 = -1 - \sqrt{5}$$

$$y_1(x) = 1, y_2(x) = x, y_3(x) = -1 + \sqrt{5}, y_4(x) = -1 - \sqrt{5}$$

Example (6) 8 The roots of aux eqn of a homo eqn with const. coeff are 0, 5, 5, 7, 7, 7, 7, $2 \pm 3i$, $2 \pm 3i$

$$Y_1(x) = e^0 = 1$$

$$Y_2(x) = e^{5x}$$

$$Y_3(x) = x e^{5x}$$

$$Y_4(x) = e^{7x}$$

$$Y_5(x) = x e^{7x}$$

$$Y_6(x) = x^2 e^{7x}$$

$$Y_7(x) = x^3 e^{7x}$$

$$Y_8(x) = e^{2x} \cos 3x$$

$$Y_9(x) = e^{2x} \sin 3x$$

$$Y_{10}(x) = x e^{2x} \cos 3x$$

$$Y_{11}(x) = x e^{2x} \sin 3x$$

the general solution is:

$$Y(x) = C_1 + e^{5x} C_2 + e^{5x} x C_3 + e^{7x} C_4 + x e^{7x} C_5 + x^2 e^{7x} C_6 + x^3 e^{7x} C_7 + e^{2x} \cos 3x C_8 + e^{2x} \sin 3x C_9 + x e^{2x} \cos 3x C_{10} + x e^{2x} \sin 3x C_{11}.$$

Rational root Theorem :

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$$

Where $a_n, a_{n-1}, \dots, a_0$ are integers the rational root to the polynomial eqn are in the form : $\frac{\text{factor of } a_0}{\text{factor of } a_n}$

example : find the general solution to the DE

$$y^{(4)} + 11y''' + 6y'' + 4y' = 0$$

Solutions are in the form $y(x) = e^{xr}$ then (r) satisfies the aux eqn :

$$r^4 + 4r^3 + 6r^2 + 4r = 0$$

$$r[r^3 + 4r^2 + 6r + 4] = 0$$

possible roots are $\pm 1, \pm 2, \pm 4$

$[r-2]$ is a factor by fail and trail.

$$\text{So } [r-2][r^2 + 2r + 2]$$

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$$\text{Now } r_1 = 0, r_2 = 2, r_3 = -1 \pm i$$

$$\text{example : } y^{(3)} - 5y'' - y' + 5y = 0$$

$$\text{aux : } r^3 - 5r^2 - r + 5 = 0$$

we try $\pm 1, \pm 5$

$$r_1 = 1, r_2 = -1, r_3 = 5 \Rightarrow y_1(x) = e^x, y_2(x) = e^{-x}, y_3(x) = e^{5x}$$

example : find the value of $f(r)$ for which the factor $y(x) = x^r$ is a solution to

$$x^3 y^{(3)} - 2x^2 y^{(2)} - 5x y' + 5y = 0$$

Cauchy Euler Order 3

$$Y(x) = x^r \Rightarrow Y'(x) = r x^{r-1} \Rightarrow Y''(x) = r(r-1) x^{r-2} \Rightarrow Y'''(x) = r(r-1)(r-2) x^{r-3}$$

Sub at the DE

$$r(r-1)(r-2) x^{r-3} - 2r(r-1) x^{r-2} - 5r x^r + 5 x^r = 0$$

$$x^r [r^3 - 3r^2 + 2r - 2r^2 + 2r - 5r + 5] = 0$$

$$x^r \neq 0, x > 0$$

$$r^3 - 5r^2 - r + 5 = 0$$

$$r_1 = 5, r_2 = 1, r_3 = -1$$

$$Y_1(x) = e^{5x} \quad Y_2(x) = e^x \quad Y_3(x) = e^{-x}$$

examples the aux eqn of 7th order homo DE With Constant Coeff is?

$$(r-1)^2 (r^2 + 7)^2 (r+6) = 0$$

$$r_1 = 1, r_2 = 1, r_3 = \pm \sqrt{7}i, r_4 = \pm \sqrt{7}i, r_5 = -6$$

$$Y_1(x) = e^x \quad Y_2(x) = x e^x \quad Y_3(x) = \cos(\sqrt{7}x) \quad Y_4(x) = \sin(\sqrt{7}x)$$

$$Y_5(x) = x \cos(\sqrt{7}x) \quad Y_6(x) = x \sin(\sqrt{7}x)$$

example: 3rd order homo DE With Constant Coeff has the solution $(e^{-x} \sin x \cos x)$

Write the DE?

$$r = -1, 0 \pm 3i \quad (r \pm 3i)^2 r$$

$$(r+1)(r-3i)(r+3i)$$

$$(r+1)(r^2 + 3ri - 3ri - 9)$$

$$(r+1)(r^2 + 9) \Rightarrow r^3 + 9r + r^2 + 9 = 0$$

$$\text{the DE eqn} \Rightarrow Y^{(3)} + Y^{(2)} + 9Y + 9 = 0$$

Method of Undetermined Coefficient:

Consider the non homogeneous DE:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = g(x), \quad g(x) \neq 0$$

To solve it, we first solve the corresponding homogeneous

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0 \quad \text{to get } y_h(x) = C_1 y_1 + C_2 y_2 + \dots + C_n y_n$$

Next we find a particular solution $y_p(x)$ to the nonhomogeneous.

$$\text{Then, } y(x) = y_h(x) + y_p(x)$$

We will use only the method of Undetermined Coeff nonhomogeneous to find

Example: Determine a suitable form of $y_p(x)$, if the method of Undetermined Coeff is used to be used:

$$(1) y^{(4)} - 4y''' + 4y'' = 3e^{2x}$$

$$\text{the corresponding homogeneous } y^{(4)} - 4y''' + 4y'' = 0$$

$$\text{has a aux } r^4 - 4r^3 + 4r^2 = 0$$

$$r^2(r^2 - 4r + 4) = 0$$

$$r^2(r-2)(r-2) = 0$$

$$r_1 = 0, r_2 = 0, r_3 = 2, r_4 = 2$$

$$y_1(x) = 1, y_2(x) = x, y_3(x) = e^{2x}, y_4(x) = x e^{2x}$$

$$\text{Now } g(x) = 3e^{2x}$$

$$\text{the initial guess is } y_p(x) = A x^2 e^{2x}$$

$$y_p''' + y_p'' + y_p' + y_p = 0 \quad \text{لأننا نبحث عن } y_p \text{ من الشكل } y_p = A x^2 e^{2x}$$

$$\text{ونفرض في المعادلة أن } y_p = A x^2 e^{2x} \text{ ونقارن المعاملات}$$

$$(2) y''' - 3y'' - 3y' - y = 4e^x$$

the corres homo equ: $y^{(3)} - 3y'' - 3y' - y = 0$

aux equ: $r^3 - 3r^2 - 3r - 1 = 0$

possible root are $r_0 = -1$ which are ± 1

by trial and fail we find that $r = 1$ is a root

So $r^3 - 3r^2 - 3r - 1 = (r-1)(r^2 - 2r + 1) = (r-1)(r-1)(r-1)$

$y_1(x) = e^x, y_2(x) = xe^x, y_3 = x^2 e^x$

Here $g(x) = 4e^x$

So initial guess for $y_p(x)$ is $y_p(x) = A x^3 e^x$

y_1, y_2, y_3 are particular solutions

$$(3) y^{(4)} + 2y'' + y = 2\sin x - 5\cos x$$

the corres homo equ: $y^{(4)} + 2y'' + y = 0$

the aux: $r^4 + 2r^2 + 1 = (r^2 + 1)(r^2 + 1) = 0$

$r_{1,2} = \pm i, r_{3,4} = \pm i$

So $y_1(x) = \cos x, y_2(x) = \sin x, y_3(x) = x \cos x, y_4(x) = x \sin x$

Now $y_p(x) = x^2 (A \sin x + B \cos x) = Ax^2 \sin x + Bx^2 \cos x$

y_1, y_2, y_3, y_4 are particular solutions

at $B \cos x$ we get $A \sin x + B \cos x = A \sin x + B \cos x$

$$(4) y''' + 9y' = x \sin 3x + x^2 e^{2x} \quad (A_1 - A_2) \sin x + (B_1 - B_2) \cos x$$

the aux: $r^3 + 9r = 0 \Rightarrow r_1 = 0, r_{2,3} = \pm 3i$

the corres equ: $y''' + 9y' = 0$

$y_1(x) = 1, y_2(x) = \cos 3x, y_3(x) = \sin 3x$

$y_p(x) = x^2 [(A_1 x + A_2) \sin 3x + (A_3 x + A_4) \cos 3x]$

$y_p(x) = e^{2x} (A_5 x^2 + A_6 x + A_7)$

$y_p = y_{p1} + y_{p2} = A_1 x^2 \sin 3x + A_2 x \sin 3x + A_3 x^2 \cos 3x + A_4 x \cos 3x + A_5 x^2 e^{2x} + A_6 x e^{2x} + A_7 e^{2x}$

Example 8 Find the general soln of $(y^{(3)} + y'' = 3e^x + 4x^2)$

the Corrs homo $y' + y'' = 0$

the aux $r^3 + r^2 = 0 \Rightarrow r^2(r+1) = 0$ ($r_{1,2} = 0$ $r_3 = -1$)

So $y_1(x) = 1$, $y_2(x) = x$, $y_3(x) = e^{-x}$

So $y_h(x) = C_1 + C_2 x + C_3 e^{-x}$

Now $g(x) = \underbrace{3e^x}_{g_1} + \underbrace{4x^2}_{g_2}$

$y_{p1} = A_1 e^x$, $y_{p2} = x^2 (A_2 x^2 + A_3 x + A_4)$ $y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9, y_{10}, y_{11}, y_{12}, y_{13}, y_{14}, y_{15}, y_{16}, y_{17}, y_{18}, y_{19}, y_{20}, y_{21}, y_{22}, y_{23}, y_{24}, y_{25}, y_{26}, y_{27}, y_{28}, y_{29}, y_{30}, y_{31}, y_{32}, y_{33}, y_{34}, y_{35}, y_{36}, y_{37}, y_{38}, y_{39}, y_{40}, y_{41}, y_{42}, y_{43}, y_{44}, y_{45}, y_{46}, y_{47}, y_{48}, y_{49}, y_{50}, y_{51}, y_{52}, y_{53}, y_{54}, y_{55}, y_{56}, y_{57}, y_{58}, y_{59}, y_{60}, y_{61}, y_{62}, y_{63}, y_{64}, y_{65}, y_{66}, y_{67}, y_{68}, y_{69}, y_{70}, y_{71}, y_{72}, y_{73}, y_{74}, y_{75}, y_{76}, y_{77}, y_{78}, y_{79}, y_{80}, y_{81}, y_{82}, y_{83}, y_{84}, y_{85}, y_{86}, y_{87}, y_{88}, y_{89}, y_{90}, y_{91}, y_{92}, y_{93}, y_{94}, y_{95}, y_{96}, y_{97}, y_{98}, y_{99}, y_{100}, y_{101}, y_{102}, y_{103}, y_{104}, y_{105}, y_{106}, y_{107}, y_{108}, y_{109}, y_{110}, y_{111}, y_{112}, y_{113}, y_{114}, y_{115}, y_{116}, y_{117}, y_{118}, y_{119}, y_{120}, y_{121}, y_{122}, y_{123}, y_{124}, y_{125}, y_{126}, y_{127}, y_{128}, y_{129}, y_{130}, y_{131}, y_{132}, y_{133}, y_{134}, y_{135}, y_{136}, y_{137}, y_{138}, y_{139}, y_{140}, y_{141}, y_{142}, y_{143}, y_{144}, y_{145}, y_{146}, y_{147}, y_{148}, y_{149}, y_{150}, y_{151}, y_{152}, y_{153}, y_{154}, y_{155}, y_{156}, y_{157}, y_{158}, y_{159}, y_{160}, y_{161}, y_{162}, y_{163}, y_{164}, y_{165}, y_{166}, y_{167}, y_{168}, y_{169}, y_{170}, y_{171}, y_{172}, y_{173}, y_{174}, y_{175}, y_{176}, y_{177}, y_{178}, y_{179}, y_{180}, y_{181}, y_{182}, y_{183}, y_{184}, y_{185}, y_{186}, y_{187}, y_{188}, y_{189}, y_{190}, y_{191}, y_{192}, y_{193}, y_{194}, y_{195}, y_{196}, y_{197}, y_{198}, y_{199}, y_{200}, y_{201}, y_{202}, y_{203}, y_{204}, y_{205}, y_{206}, y_{207}, y_{208}, y_{209}, y_{210}, y_{211}, y_{212}, y_{213}, y_{214}, y_{215}, y_{216}, y_{217}, y_{218}, y_{219}, y_{220}, y_{221}, y_{222}, y_{223}, y_{224}, y_{225}, y_{226}, y_{227}, y_{228}, y_{229}, y_{230}, y_{231}, y_{232}, y_{233}, y_{234}, y_{235}, y_{236}, y_{237}, y_{238}, y_{239}, y_{240}, y_{241}, y_{242}, y_{243}, y_{244}, y_{245}, y_{246}, y_{247}, y_{248}, y_{249}, y_{250}, y_{251}, y_{252}, y_{253}, y_{254}, y_{255}, y_{256}, y_{257}, y_{258}, y_{259}, y_{260}, y_{261}, y_{262}, y_{263}, y_{264}, y_{265}, y_{266}, y_{267}, y_{268}, y_{269}, y_{270}, y_{271}, y_{272}, y_{273}, y_{274}, y_{275}, y_{276}, y_{277}, y_{278}, y_{279}, y_{280}, y_{281}, y_{282}, y_{283}, y_{284}, y_{285}, y_{286}, y_{287}, y_{288}, y_{289}, y_{290}, y_{291}, y_{292}, y_{293}, y_{294}, y_{295}, y_{296}, y_{297}, y_{298}, y_{299}, y_{300}, y_{301}, y_{302}, y_{303}, y_{304}, y_{305}, y_{306}, y_{307}, y_{308}, y_{309}, y_{310}, y_{311}, y_{312}, y_{313}, y_{314}, y_{315}, y_{316}, y_{317}, y_{318}, y_{319}, y_{320}, y_{321}, y_{322}, y_{323}, y_{324}, y_{325}, y_{326}, y_{327}, y_{328}, y_{329}, y_{330}, y_{331}, y_{332}, y_{333}, y_{334}, y_{335}, y_{336}, y_{337}, y_{338}, y_{339}, y_{340}, y_{341}, y_{342}, y_{343}, y_{344}, y_{345}, y_{346}, y_{347}, y_{348}, y_{349}, y_{350}, y_{351}, y_{352}, y_{353}, y_{354}, y_{355}, y_{356}, y_{357}, y_{358}, y_{359}, y_{360}, y_{361}, y_{362}, y_{363}, y_{364}, y_{365}, y_{366}, y_{367}, y_{368}, y_{369}, y_{370}, y_{371}, y_{372}, y_{373}, y_{374}, y_{375}, y_{376}, y_{377}, y_{378}, y_{379}, y_{380}, y_{381}, y_{382}, y_{383}, y_{384}, y_{385}, y_{386}, y_{387}, y_{388}, y_{389}, y_{390}, y_{391}, y_{392}, y_{393}, y_{394}, y_{395}, y_{396}, y_{397}, y_{398}, y_{399}, y_{400}, y_{401}, y_{402}, y_{403}, y_{404}, y_{405}, y_{406}, y_{407}, y_{408}, y_{409}, y_{410}, y_{411}, y_{412}, y_{413}, y_{414}, y_{415}, y_{416}, y_{417}, y_{418}, y_{419}, y_{420}, y_{421}, y_{422}, y_{423}, y_{424}, y_{425}, y_{426}, y_{427}, y_{428}, y_{429}, y_{430}, y_{431}, y_{432}, y_{433}, y_{434}, y_{435}, y_{436}, y_{437}, y_{438}, y_{439}, y_{440}, y_{441}, y_{442}, y_{443}, y_{444}, y_{445}, y_{446}, y_{447}, y_{448}, y_{449}, y_{450}, y_{451}, y_{452}, y_{453}, y_{454}, y_{455}, y_{456}, y_{457}, y_{458}, y_{459}, y_{460}, y_{461}, y_{462}, y_{463}, y_{464}, y_{465}, y_{466}, y_{467}, y_{468}, y_{469}, y_{470}, y_{471}, y_{472}, y_{473}, y_{474}, y_{475}, y_{476}, y_{477}, y_{478}, y_{479}, y_{480}, y_{481}, y_{482}, y_{483}, y_{484}, y_{485}, y_{486}, y_{487}, y_{488}, y_{489}, y_{490}, y_{491}, y_{492}, y_{493}, y_{494}, y_{495}, y_{496}, y_{497}, y_{498}, y_{499}, y_{500}, y_{501}, y_{502}, y_{503}, y_{504}, y_{505}, y_{506}, y_{507}, y_{508}, y_{509}, y_{510}, y_{511}, y_{512}, y_{513}, y_{514}, y_{515}, y_{516}, y_{517}, y_{518}, y_{519}, y_{520}, y_{521}, y_{522}, y_{523}, y_{524}, y_{525}, y_{526}, y_{527}, y_{528}, y_{529}, y_{530}, y_{531}, y_{532}, y_{533}, y_{534}, y_{535}, y_{536}, y_{537}, y_{538}, y_{539}, y_{540}, y_{541}, y_{542}, y_{543}, y_{544}, y_{545}, y_{546}, y_{547}, y_{548}, y_{549}, y_{550}, y_{551}, y_{552}, y_{553}, y_{554}, y_{555}, y_{556}, y_{557}, y_{558}, y_{559}, y_{560}, y_{561}, y_{562}, y_{563}, y_{564}, y_{565}, y_{566}, y_{567}, y_{568}, y_{569}, y_{570}, y_{571}, y_{572}, y_{573}, y_{574}, y_{575}, y_{576}, y_{577}, y_{578}, y_{579}, y_{580}, y_{581}, y_{582}, y_{583}, y_{584}, y_{585}, y_{586}, y_{587}, y_{588}, y_{589}, y_{590}, y_{591}, y_{592}, y_{593}, y_{594}, y_{595}, y_{596}, y_{597}, y_{598}, y_{599}, y_{600}, y_{601}, y_{602}, y_{603}, y_{604}, y_{605}, y_{606}, y_{607}, y_{608}, y_{609}, y_{610}, y_{611}, y_{612}, y_{613}, y_{614}, y_{615}, y_{616}, y_{617}, y_{618}, y_{619}, y_{620}, y_{621}, y_{622}, y_{623}, y_{624}, y_{625}, y_{626}, y_{627}, y_{628}, y_{629}, y_{630}, y_{631}, y_{632}, y_{633}, y_{634}, y_{635}, y_{636}, y_{637}, y_{638}, y_{639}, y_{640}, y_{641}, y_{642}, y_{643}, y_{644}, y_{645}, y_{646}, y_{647}, y_{648}, y_{649}, y_{650}, y_{651}, y_{652}, y_{653}, y_{654}, y_{655}, y_{656}, y_{657}, y_{658}, y_{659}, y_{660}, y_{661}, y_{662}, y_{663}, y_{664}, y_{665}, y_{666}, y_{667}, y_{668}, y_{669}, y_{670}, y_{671}, y_{672}, y_{673}, y_{674}, y_{675}, y_{676}, y_{677}, y_{678}, y_{679}, y_{680}, y_{681}, y_{682}, y_{683}, y_{684}, y_{685}, y_{686}, y_{687}, y_{688}, y_{689}, y_{690}, y_{691}, y_{692}, y_{693}, y_{694}, y_{695}, y_{696}, y_{697}, y_{698}, y_{699}, y_{700}, y_{701}, y_{702}, y_{703}, y_{704}, y_{705}, y_{706}, y_{707}, y_{708}, y_{709}, y_{710}, y_{711}, y_{712}, y_{713}, y_{714}, y_{715}, y_{716}, y_{717}, y_{718}, y_{719}, y_{720}, y_{721}, y_{722}, y_{723}, y_{724}, y_{725}, y_{726}, y_{727}, y_{728}, y_{729}, y_{730}, y_{731}, y_{732}, y_{733}, y_{734}, y_{735}, y_{736}, y_{737}, y_{738}, y_{739}, y_{740}, y_{741}, y_{742}, y_{743}, y_{744}, y_{745}, y_{746}, y_{747}, y_{748}, y_{749}, y_{750}, y_{751}, y_{752}, y_{753}, y_{754}, y_{755}, y_{756}, y_{757}, y_{758}, y_{759}, y_{760}, y_{761}, y_{762}, y_{763}, y_{764}, y_{765}, y_{766}, y_{767}, y_{768}, y_{769}, y_{770}, y_{771}, y_{772}, y_{773}, y_{774}, y_{775}, y_{776}, y_{777}, y_{778}, y_{779}, y_{780}, y_{781}, y_{782}, y_{783}, y_{784}, y_{785}, y_{786}, y_{787}, y_{788}, y_{789}, y_{790}, y_{791}, y_{792}, y_{793}, y_{794}, y_{795}, y_{796}, y_{797}, y_{798}, y_{799}, y_{800}, y_{801}, y_{802}, y_{803}, y_{804}, y_{805}, y_{806}, y_{807}, y_{808}, y_{809}, y_{810}, y_{811}, y_{812}, y_{813}, y_{814}, y_{815}, y_{816}, y_{817}, y_{818}, y_{819}, y_{820}, y_{821}, y_{822}, y_{823}, y_{824}, y_{825}, y_{826}, y_{827}, y_{828}, y_{829}, y_{830}, y_{831}, y_{832}, y_{833}, y_{834}, y_{835}, y_{836}, y_{837}, y_{838}, y_{839}, y_{840}, y_{841}, y_{842}, y_{843}, y_{844}, y_{845}, y_{846}, y_{847}, y_{848}, y_{849}, y_{850}, y_{851}, y_{852}, y_{853}, y_{854}, y_{855}, y_{856}, y_{857}, y_{858}, y_{859}, y_{860}, y_{861}, y_{862}, y_{863}, y_{864}, y_{865}, y_{866}, y_{867}, y_{868}, y_{869}, y_{870}, y_{871}, y_{872}, y_{873}, y_{874}, y_{875}, y_{876}, y_{877}, y_{878}, y_{879}, y_{880}, y_{881}, y_{882}, y_{883}, y_{884}, y_{885}, y_{886}, y_{887}, y_{888}, y_{889}, y_{890}, y_{891}, y_{892}, y_{893}, y_{894}, y_{895}, 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$y_p = y_{p1} + y_{p2} = A_1 e^x + A_2 x + A_3 x^3 + A_4 x^2$

To find A_1, A_2, A_3, A_4

$y'p(x) = A_1 e^x + A_2 4x^3 + 3A_3 x^2 + 2A_4 x$

$y''p(x) = A_1 e^x + 12x^2 A_2 + 6x A_3$

$y'''p(x) = A_1 e^x + 24x A_2 + 6A_3$

Sub in the DE: $A_1 e^x + 24A_2 x + 6A_3 + A_1 e^x + 12A_2 x^2 + 6A_3 x + 2A_4 = 3e^x + 4x^2$
 $2A_1 e^x + 12A_2 x^2 + (24A_2 + 6A_3)x + (6A_3 + 2A_4) = 3e^x + 4x^2$

$e^x: 2A_1 = 3 \Rightarrow A_1 = 3/2$

$x^2: 12A_2 = 4 \Rightarrow A_2 = 1/3$

$x: 24A_2 + 6A_3 = 0 \Rightarrow A_3 = -4/3$

$x^0: 6A_3 + 2A_4 = 0 \Rightarrow A_4 = 4$

CH5 Laplace transform.

Given a function $f(t)$, $t \geq 0$, the Laplace transform of $f(t)$ is denoted by $L[f(t)] = \mathcal{L}[f(t)]$ and defined by:

$$\mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} f(t) dt, \text{ this is an improper integral}$$

$$\text{So } \mathcal{L}[f(t)] = \lim_{a \rightarrow \infty} \int_0^a e^{-st} f(t) dt \begin{cases} \text{exist (Conv)} \\ \text{d.n.e (div)} \end{cases}$$

$$\mathcal{L}[f(t)] \xrightarrow[\text{جواب}]{\text{Function}} F(s)$$

example: Find $\mathcal{L}[e^{2t}]$?

$$\begin{aligned} \mathcal{L}[e^{2t}] &= \int_0^{\infty} e^{-st} \cdot e^{2t} dt \\ &= \lim_{a \rightarrow \infty} \int_0^a e^{-(s-2)t} dt \\ &= \lim_{a \rightarrow \infty} \left[\frac{e^{-(s-2)t}}{-(s-2)} \right]_0^a = \lim_{a \rightarrow \infty} \left(\frac{e^{-(s-2)a} - 1}{-(s-2)} \right) = \frac{1}{s-2}, s > 2 \end{aligned}$$

example: $\mathcal{L}[t]$?

$$\begin{aligned} \mathcal{L}[t] &= \int_0^{\infty} e^{-st} t dt \\ &= \lim_{a \rightarrow \infty} \int_0^a e^{-st} t dt = \lim_{a \rightarrow \infty} \left[-\frac{t e^{-st}}{s} - \int \frac{e^{-st}}{-s} dt \right]_0^a \\ &= \lim_{a \rightarrow \infty} \left(-\frac{t e^{-st}}{s} - \frac{e^{-st}}{s^2} \right) \Big|_0^a \\ &= \lim_{a \rightarrow \infty} \left(-\frac{a e^{-as}}{s} - \frac{e^{-as}}{s^2} \right) + \frac{1}{s^2} \\ &= \frac{1}{s^2}, s > 0 \end{aligned}$$

example: let $f(x) = \begin{cases} 2, & 0 \leq t < 1 \\ 4, & t \geq 1 \end{cases}$ find $\mathcal{L}[f(t)]$?

$$\mathcal{L}[f(t)] = \int_0^1 2e^{-st} dt + \int_1^{\infty} 4e^{-st} dt$$

$$= \left. \frac{2e^{-st}}{-s} \right|_0^1 + \lim_{a \rightarrow \infty} \int_1^a 4e^{-st} dt$$

$$= \frac{-2}{s} (e^{-s} - 1) + \lim_{a \rightarrow \infty} \frac{-4}{s} (e^{-as} - e^{-s}) \quad \sim s > 0$$

$$= \frac{-2}{s} (e^{-s} - 1) + \frac{4}{s} e^{-s}$$

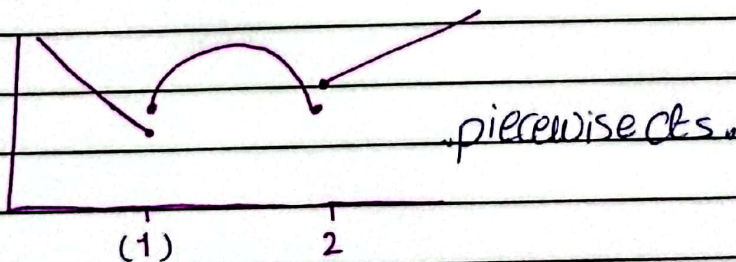
$$= \frac{2}{s} e^{-s} + \frac{2}{s} *$$

Theorem: Suppose that $f(t)$ is piecewise cts on the interval $0 \leq t \leq A$, and $|f(t)| \leq Ke^{at}$ when $t \geq A$ then $\mathcal{L}[f(t)]$ exist for $s > a$.

يعني: كل دالة تكون متصلة على الفترة $[0, A]$ (بشروط معينة) ولها شكل $|f(t)| \leq Ke^{at}$ عندما $t \geq A$ فإن $\mathcal{L}[f(t)]$ موجودة لـ $s > a$.

وذلك لأن: $\lim_{t \rightarrow A^-} f(t) = \text{exist} \neq \lim_{t \rightarrow A^+} f(t) = \text{exist}$

ase



Jump discts point



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دفتر :

معادلات تفاضلية عادية

Ordinary Differential Equations Lab

الاسبوع الحادي عشر

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اللجنة الأكاديمية لقسم الهندسة الصناعية

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Example: find?

$$1) \mathcal{L}^{-1} \left[\frac{1}{(s-2)^4} \right] = \frac{1}{6} e^{2t} t^3$$

$$2) \mathcal{L}^{-1} \left[\frac{1}{s^2 - 6s} \right] = \frac{1}{3} e^{3t} \sinh 3t$$

Using Partial Fractions To find $\mathcal{L}^{-1} [F(s)]$?

To illustrate the idea, we will start with this example:

$$\mathcal{L}^{-1} \left[\frac{1}{s^3 - s} \right] = \frac{1}{s(s^2 - 1)} = \frac{1}{s(s-1)(s+1)}$$

$$1 = \frac{A(s-1)(s+1)}{(s+1)s(s-1)} + \frac{B s(s+1)}{s-1(s)(s+1)} + \frac{C s(s-1)}{s+1 s(s-1)}$$

$$1 = A(s-1)^2 + B(s+1)(s) + C(s)(s-1)$$

$$s=0 \rightarrow 1 = -A \Rightarrow A = -1$$

$$s=1 \Rightarrow 1 = 2B \Rightarrow B = 1/2$$

$$s=-1 \Rightarrow 1 = 2C \Rightarrow C = 1/2$$

became: $\frac{1}{s^3 - s} = \frac{-1}{s} + \frac{1/2}{s-1} + \frac{1/2}{s+1}$ $\mathcal{L}^{-1}$ izi

$$\mathcal{L}^{-1} \frac{1}{s^3 - s} = \mathcal{L}^{-1} \left[\frac{-1}{s} \right] + \mathcal{L}^{-1} \left[\frac{1/2}{s-1} \right] + \mathcal{L}^{-1} \left[\frac{1/2}{s+1} \right]$$

$$= -1 + 1/2 e^t + 1/2 e^{-t}$$

Example: $\mathcal{L}[e^{3t} - t + 1]$?

$$= \mathcal{L}[e^{3t}] - \mathcal{L}[t] + \mathcal{L}[1]$$

$$= \frac{1}{s-3} - \frac{1}{s^2} + \frac{1}{s}$$

Example: $\mathcal{L}[\sinh 3t]$

$$= \mathcal{L}\left[\frac{e^{3t} - e^{-3t}}{2}\right] = \frac{1}{2} (\mathcal{L}[e^{3t}] - \mathcal{L}[e^{-3t}])$$

$f(x)$	$\mathcal{L}[f(t)]$
C	$\frac{C}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$

example: $\mathcal{L}[\sin t] = \mathcal{L}[\frac{1}{2}(1 - \cos t)]$

$$= \frac{1}{2} (\mathcal{L}[1] - \mathcal{L}[\cos t])$$

$$= \frac{1}{2} \left(\frac{1}{s} - \frac{s}{s^2+a^2} \right)$$

example: $\mathcal{L}[\cos t] = \mathcal{L}[\frac{1}{2}(1 + \cos t)]$

$$= \frac{1}{2} \left(\frac{1}{s} + \frac{s}{s^2-a^2} \right) = \frac{3}{s^2-a^2}$$

(2) Shifting property:

if $\mathcal{L}[f(t)] = F(s)$ Then $\mathcal{L}[e^{at} f(t)] = F(s-a)$

example: $\mathcal{L}[e^{2t} \sin 3t] = \frac{3}{(s-2)^2 + 9}$

example: $\mathcal{L}[e^{-t} \cos(2t)] = \frac{s+1}{s^2 + 4}$ $s \rightarrow s-a$
 $s \rightarrow s-(-1)$
 $(s+1)$

example: $\mathcal{L}[t^2 e^{-3t}] = \frac{2}{(s+3)^3}$ $s \rightarrow s-(-3) \rightarrow s+3$

example: $\mathcal{L}[e^{2t} \cos^2(4t)] = \frac{1}{2} \left(\frac{1}{s-2} + \frac{s-2}{(s-2)^2 + 64} \right)$

example: $\mathcal{L}[\cos(4t)] = \mathcal{L}\left[\frac{1}{2} (1 + \cos 8t)\right] = \frac{1}{2} \left(\frac{1}{s} + \frac{s}{s^2 + 64} \right)$

(3) Multiplication by t

if $\mathcal{L}[f(t)] = F(s)$ then $\mathcal{L}[t^n f(t)] = (-1)^n F^{(n)}(s)$

example: evaluate $\mathcal{L}[t' \cos 2t]$?

$$= (-1)' \left(\frac{s}{s^2 + 4} \right)' = (-1) \frac{(s^2 + 4)(1) - s(2s)}{(s^2 + 4)^2}$$

$$= \frac{s^2 - 4}{(s^2 + 4)^2}$$

example: evaluate $\mathcal{L}[t^2 \sin t]$?

$$= (-1)^2 \left(\frac{1}{s^2 + 1} \right)'' = \left(\frac{-2s}{(s^2 + 1)^2} \right)'$$

$$= -2 \left(\frac{(s^2 + 1)^2(1) - 2s(s^2 + 1)(2s)}{(s^2 + 1)^4} \right)$$

$$= \frac{6s^2 - 2}{(s^2 + 1)^3}$$

Example 8 evaluate $\mathcal{L}[t^2 e^{3t}]$?

Shifting $\mathcal{L}[t^2 e^{3t}] = \frac{2}{(s-3)^3}$

Using multipling $\Rightarrow \mathcal{L}[t^2 e^{3t}] = (-1)^2 \left(\frac{1}{s-3} \right)''$
 $= (s-3)^{-1}''$
 $= (\ominus(s-3)^{-2})' = 2(s-3)^{-3} = \frac{2}{(s-3)^3}$

Example 8 evaluate $\mathcal{L}[t \sin t] = (-1)' \left(\frac{1}{2} \left(\frac{1}{s} - \frac{s}{s^2+4} \right) \right)$

$\mathcal{L}(\sin^2 t) = \mathcal{L}\left[\frac{1}{2}(1 - \cos 2t)\right] = \frac{-1}{2} \left(\frac{1}{s} - \frac{s}{s^2+4} \right)'$
 $= \frac{1}{2} \left(\frac{1}{s} - \frac{s}{s^2+4} \right)$
 $= \frac{-1}{2} \left(-\frac{1}{s^2} - \frac{(s^2+4)(1) - s(2s)}{(s^2+4)^2} \right)$
 $= \frac{1}{2} \left(\frac{1}{s^2} + \frac{4-s^2}{(s^2+4)^2} \right)$

example 8 evaluate $\mathcal{L}[t e^{2t} \sin 5t] = \mathcal{L}[e^{2t} t \sin 5t]$

$\mathcal{L}[t \sin 5t] = (-1)' \left(\frac{5}{s^2+25} \right)'$
 $= -5 \frac{(-2s)}{(s^2+25)^2} = \frac{10s}{(s^2+25)^2}$

$\mathcal{L}[e^{2t} t \sin 5t] = \frac{10(s-2)}{((s-2)^2+25)^2}$

• If $\mathcal{L}[f(x)] = F(s)$ then $\lim_{s \rightarrow \infty} F(s) = 0$

$$\frac{1}{s^2 + sa}$$

$$\frac{s^2}{s^3 + 7}$$

$$\frac{s^2}{s^2 + 1}$$

• let $f(t), f'(t), \dots, f^{(n-1)}(t)$ be cts for $t \geq 0$ and let $f^{(n)}(t)$ be piecewise cts for $t \geq 0$ and $\mathcal{L}(f^{(n)}(t)) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0)$

- Special Cases

$$\mathcal{L}[f'(t)] = sF(s) - f(0)$$

$$\mathcal{L}[f''(t)] = s^2 F(s) - sf(0) - f'(0)$$

$$\mathcal{L}[f'''(t)] = s^3 F(s) - s^2 f(0) - sf'(0) - f''(0)$$

example: $y'' + 2y' + y = \sin t$; $y(0) = 1$; $y'(0) = 0$

Along let $y(t) \rightarrow \mathcal{L}[y(t)] = Y(s)$

$$\mathcal{L}[y'' + 2y' + y] = \mathcal{L}[\sin t]$$

$$\mathcal{L}(y'') + 2\mathcal{L}(y') + \mathcal{L}(y) = \mathcal{L}[\sin t]$$

$$s^2 Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + Y(s) = \frac{1}{s^2 + 1}$$

$$[s^2 Y(s) - s + 2sY(s) - 2 + Y(s)] = \frac{1}{s^2 + 1}$$

$$(s^2 + 2s + 1)Y(s) = \frac{1}{s^2 + 1} + s + 2$$

$$Y(s) = \frac{1}{(s^2 + 1)(s^2 + 2s + 1)} + \frac{s + 2}{s^2 + 2s + 1}$$

But we want $y(t) = \mathcal{L}^{-1}(Y(s))$ ($Y(s)$ is given we want $y(t)$)!

• Laplace inverse:

If $\mathcal{L}[f(t)] = F(s)$, then the operator $(\mathcal{L}^{-1})$ called Laplace inverse works as follow $\mathcal{L}^{-1}[F(s)] = f(t)$

$\mathcal{L}^{-1}[f(s)]$ We think of this problem in this way we seek a function $F(t)$ such that $\mathcal{L}[F(t)] = f(s)$

example 3 Find laplace inverse in the following :

$$(1) \mathcal{L}^{-1}\left(\frac{7}{s}\right) = 7$$

$$(2) \mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right] = \frac{1}{2} \sin 2t \quad \mathcal{L}[\sin 2t] = \frac{2}{s^2+4}$$

$$\frac{1}{2} \mathcal{L}[\sin 2t] = \frac{1}{s^2+4} \Rightarrow \mathcal{L}\left[\frac{1}{2} \sin 2t\right] = \frac{1}{s^2+4}$$

$$(3) \mathcal{L}^{-1}\left[\frac{1}{s+4}\right] = \mathcal{L}^{-1}[e^{-4t}] = \frac{1}{s+4}$$

$$(4) \mathcal{L}^{-1}\left[\frac{1}{s^3}\right] = \frac{1}{2} t^2 \quad \mathcal{L}[t^2] = \frac{2}{s^3} = \mathcal{L}\left[\frac{1}{2} t^2\right] = \frac{1}{s^3}$$

$$(5) \mathcal{L}^{-1}\left[\frac{3}{s^4}\right] = \frac{1}{2} t^3 \quad \mathcal{L}[t^3] = \frac{6}{s^4}$$

$$\mathcal{L}\left[\frac{1}{2} t^3\right] = \frac{3}{s^4}$$

$$(6) \mathcal{L}^{-1}\left[\frac{s}{s^2+10}\right] = \cos \sqrt{10} t \quad \mathcal{L}[\cos \sqrt{10} t] = \frac{s}{s^2+10}$$

$$(7) \mathcal{L}^{-1}\left[\frac{1}{s^2-2}\right] = \frac{1}{\sqrt{2}} \sinh \sqrt{2} t \quad \mathcal{L}[\sinh \sqrt{2} t] = \frac{\sqrt{2}}{s^2-2}$$

$$\mathcal{L}\left[\frac{1}{\sqrt{2}} \sinh \sqrt{2} t\right] = \frac{1}{s^2-2}$$

$$(8) \mathcal{L}^{-1}\left[\frac{1}{(s-2)^3}\right] = \frac{1}{2} e^{2t} t^2$$

$$\mathcal{L}(t^2) = \frac{2}{s^3}$$

$$\mathcal{L}[e^{2t} t^2] = \frac{2}{(s-2)^3}$$

$$\mathcal{L}\left[\frac{1}{2} e^{2t} t^2\right] = \frac{1}{(s-2)^3}$$

$$(9) \mathcal{L}^{-1} \left[\frac{1}{s^2 + 4s + 6} \right] = \mathcal{L}^{-1} \left[\frac{1}{(s+2)^2 + 2} \right]$$

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$$\begin{aligned} s^2 + 4s + 6 &= \left(\frac{s+2}{2} \right)^2 \\ s^2 + 4s + 6 &= s^2 + 4s + 6 + 4 - 4 \\ &= s^2 + 4s + 4 + 2 \\ &= (s+2)^2 + 2. \end{aligned}$$

$$\mathcal{L} [\sin \sqrt{2} t] = \frac{\sqrt{2}}{s^2 + 2}$$

$$\mathcal{L} [e^{-2t} \sin \sqrt{2} t] = \frac{\sqrt{2}}{(s+2)^2 + 2}$$

$$\mathcal{L} \left[\frac{1}{2} e^{-2t} \sin \sqrt{2} t \right] = \frac{1}{(s+2)^2 + 2}$$

$$(10) \mathcal{L}^{-1} \left[\frac{1}{s^2 - s} \right] = 2 e^{\frac{1}{2}t} \sinh \frac{1}{2} t$$

$$s^2 - s = s^2 - s + \frac{1}{4} - \frac{1}{4} \Rightarrow \left(s - \frac{1}{2} \right)^2 - \frac{1}{4}$$

$$\mathcal{L}^{-1} \left[\frac{1}{(s - \frac{1}{2})^2 - \frac{1}{4}} \right] = \mathcal{L} [\sinh \frac{1}{2} t] = \frac{1/2}{s^2 - 1/4}$$

$$= \mathcal{L} [e^{\frac{1}{2}t} \sinh \frac{1}{2} t] = \frac{1/2}{(s - 1/2)^2 - \frac{1}{4}}$$

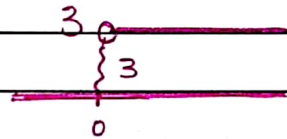
$$= \mathcal{L} [2e^{\frac{1}{2}t} \sinh \frac{1}{2} t] = \frac{1}{(s - 1/2)^2 - \frac{1}{4}}$$

Transforms of discontinuous Functions

Def: the function $u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$ is called the Unit Step function



$$3u(t) = \begin{cases} 0, & t < 0 \\ 3, & t > 0 \end{cases}$$



We can make a shift: $u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$

" $u(t-a)$ is called the Unit Step function at (a) ."

$$\mathcal{L}[u(t-a)] = \int_0^{\infty} e^{-st} u(t-a) dt$$

$$= \int_0^a e^{-st} u(t-a) dt + \int_a^{\infty} e^{-st} u(t-a) dt$$

$$= \int_a^{\infty} e^{-st} dt = \lim_{b \rightarrow \infty} \int_a^b e^{-st} dt = \lim_{b \rightarrow \infty} \left[\frac{e^{-st}}{-s} \right]_a^b$$

$$= \lim_{b \rightarrow \infty} \frac{e^{-bs} - e^{-as}}{-s} = \frac{e^{-as}}{s}$$

$$\mathcal{L}[u(t-a)] = \frac{e^{-as}}{s}$$

For example: $\mathcal{L}[u(t-3)] = \frac{e^{-3s}}{s}$

$$\mathcal{L}[u(t-\pi)] = \frac{e^{-\pi s}}{s}$$

- Any discontinuous function with finite number of jump discts can be expressed as a comp of Unit Step function.

Example 8 Express the given function using step function and compute its laplace transform.

$$(1) f(t) = \begin{cases} 1 & 0 < t < 1 \\ -2 & 1 < t < 2 \\ 3 & t > 2 \end{cases}$$

$$f(t) = 1 - 3u(t-1) + 5u(t-2)$$

$$\begin{aligned} \mathcal{L}[f(t)] &= \mathcal{L}[1] - 3\mathcal{L}[u(t-1)] + 5\mathcal{L}[u(t-2)] \\ &= \frac{1}{s} - 3\frac{e^{-s}}{s} + 5\frac{e^{-2s}}{s} \end{aligned}$$

$$(2) g(t) = \begin{cases} 3 & t < 2 \\ 1 & 2 < t < 5 \\ 10 & 5 < t < 8 \\ 7 & t > 8 \end{cases}$$

$$g(t) = 3 - 2u(t-2) + 9u(t-5) - 3u(t-8)$$

$$\begin{aligned} \mathcal{L}[g(t)] &= \mathcal{L}[3] - 2\mathcal{L}[u(t-2)] + 9\mathcal{L}[u(t-5)] - 3\mathcal{L}[u(t-8)] \\ &= \frac{3}{s} - 2\frac{e^{-2s}}{s} + 9\frac{e^{-5s}}{s} - 3\frac{e^{-8s}}{s} \end{aligned}$$

We need also to know that :

$$\mathcal{L}[g(t)u(t-a)] = e^{-as} \mathcal{L}[g(t+a)]$$

$$\mathcal{L}^{-1}[e^{-as}F(s)] = f(t-a)u(t-a)$$

Example 9 Compute :

$$\begin{aligned} (1) \mathcal{L}[t u(t-3)] &= e^{-3s} \mathcal{L}[t+3] \\ &= e^{-3s} \left(\frac{1}{s^2} + \frac{3}{s} \right) \end{aligned}$$

$$\begin{aligned}
 (2) \mathcal{L}[t^2 u(t-1)] &= e^{-s} \mathcal{L}[(t+1)^2] \\
 &= e^{-s} \mathcal{L}[t^2 + 2t + 1] \\
 &= e^{-s} \left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s} \right)
 \end{aligned}$$

$$\begin{aligned}
 (3) \mathcal{L}[(t+1) u(t-2)] &= e^{-2s} \mathcal{L}[t+3] \\
 &= e^{-2s} \left(\frac{1}{s^2} + \frac{3}{s} \right)
 \end{aligned}$$

$$(4) \mathcal{L}[f(t)] \text{ where } f(t) = \begin{cases} 1 & , 0 < t < 2 \\ t+2 & , t \geq 2 \end{cases}$$

$$f(t) = 1 + (t+1)u(t-2)$$

$$\mathcal{L}[f(t)] = \mathcal{L}[1] + \mathcal{L}[(t+1)u(t-2)] = \frac{1}{s} + e^{-2s} \left(\frac{1}{s^2} + \frac{3}{s} \right)$$

• example find $\mathcal{L}^{-1} \left[\frac{e^{-2s}}{s^2+1} \right] = \sin(t-2) u(t-2)$

$$\mathcal{L}^{-1} \left[\frac{e^{-3s}}{s^4} \right] = \frac{1}{6} (t-3)^3 u(t-3)$$

• Compute the laplace transform in the following :

$$\begin{aligned}
 (1) \mathcal{L}[e^{3t} u(t-4)] &= e^{-4s} \mathcal{L}[e^{3t} u(t-4)] \\
 &= e^{-4s} \mathcal{L}[e^{3(t-4)}] \\
 &= e^{-4s} \mathcal{L}[e^{3t-12}] \\
 &= e^{-4s} \mathcal{L}[e^{12} \cdot e^{3t}] \\
 &= e^{12} e^{-4s} \mathcal{L}[e^{3t}]
 \end{aligned}$$

Now $\mathcal{L}[f(x) u(t-a)] = e^{-as} \mathcal{L}[f(t+a)]$

$$\mathcal{L}^{-1}[e^{as} F(s)] = u(t-a) f(t-a)$$

$$\mathcal{L}^{-1}[e^{9s}] = \frac{e^{-9s}}{s}$$

$$\bullet \mathcal{L}^{-1} \left[\frac{e^{-3s}}{s^2} \right] = u(t-3) f(t-3) \quad \frac{1}{s^2} = f(s)$$

$$\mathcal{L}^{-1} [f(s)] = \mathcal{L}^{-1} \left[\frac{1}{s^2} \right]$$

$$= t \quad (u(t-3)(t-3))$$

$$\bullet \mathcal{L}^{-1} \left[\frac{e^{-2s}}{s^2+a} \right] = u(t-2) f(t-2) \quad F(s) = \frac{1}{s^2+a}$$

$$= \frac{1}{3} \sin(3(t-2) u(t-2)) \quad = \frac{\sin(3t)}{3}$$

$$\bullet \mathcal{L}^{-1} \left[\frac{s e^{-3s}}{s^2+4s+5} \right] = u(t-3) f(t-3)$$

$$s^2+4s+5 =$$

$$s^2+4s+4-4+5 = (s+2)^2+1$$

$$\mathcal{L}^{-1} \left[\frac{s+2-2}{(s+2)^2+1} \right] = \frac{s+2}{(s+2)^2+1} - 2 \mathcal{L}^{-1} \left[\frac{1}{(s+2)^2+1} \right]$$

Shift

$$= e^{-2t} \cos t - 2 e^{-2t} \sin t \quad (t-3)$$

$$= (t-3) e^{-2(t-3)} \cos(t-3) - 2 e^{-2(t-3)} \sin(t-3)$$

$$\bullet \mathcal{L}^{-1} \left[\frac{e^{-s}(s-3)}{(s+1)(s+2)} \right] = u(t-1) f(t-1) \quad \text{نقل الوقت}$$

$$= \mathcal{L}^{-1} \left[\frac{(s-3)}{(s+1)(s+2)} \right] \quad \text{البحث عن كسور جزئية}$$

$$\mathcal{L}^{-1} \left[\frac{-4}{s+1} + \frac{5}{s+2} \right] \quad \left| \begin{array}{l} (s-3) = \frac{A}{s+1} + \frac{B}{s+2} \\ (A=-4 \quad B=5) \end{array} \right.$$

$$\rightarrow -4e^{-t} + 5e^{-2t}$$

$$-4e^{-(t-1)} + 5e^{-2(t-1)} = -4e^{(1-t)} + 5e^{(2-2t)}$$

$$\bullet \mathcal{L}^{-1} \left[\frac{e^{-2s}}{s+3} - 3e^{-4s} \right] = \left[\frac{e^{-2s}}{s+3} \right] \mathcal{L}^{-1} \left[\frac{3e^{-4s}}{s+3} \right]$$

$$u(t-2) \frac{1}{s+3} \quad u(t-4) \frac{3}{s+3} \quad \left\{ \begin{array}{l} 0 < t < 2 \\ e^{-3(t-2)} & t > 2 \end{array} \right.$$

$$u(t-2) e^{-3(t-2)} - 3u(t-4) e^{-3(t-4)}$$

$$\begin{array}{c} 0 \quad 2 \quad 4 \\ \hline 0 \quad e^{-3(t-2)} \quad e^{-3(t-4)} + -3e^{-3(t-4)} \end{array}$$

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$$\left\{ \begin{array}{l} 0 < t < 4 \\ -3e^{-3(t-4)} & t > 4 \end{array} \right.$$



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$$\left\{ \begin{array}{l} 0 \leq t < 2 \\ e^{-\lambda(t-2)}, \quad 2 < t < 4 \\ e^{-\lambda(t-2)} - 3e^{-\lambda(t-4)}, \quad t > 4 \end{array} \right\}$$

Solving IVP By Laplace Transform's Consider an IVP:

$$Y' + ay = f(t)$$

$$Y(0) = \alpha$$

Recall that $Y' + ay = f(t)$ (Apply the Laplace to the DE)

$$\mathcal{L}[Y' + ay] = \mathcal{L}[f(t)]$$

$$\mathcal{L}[Y'] + a\mathcal{L}[Y] = F(s)$$

$$sY(s) + aY(s) - Y(0) = F(s)$$

$$(s+a)Y(s) = F(s) + Y(0)$$

$$Y(s) = \frac{F(s) + Y(0)}{(s+a)}$$

$$\text{But } \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{F(s) + Y(0)}{s+a}\right]$$

$$\square \mathcal{L}[Y'(t)] = sY(s) - Y(0)$$

$$\square \mathcal{L}[Y''(t)] = s^2Y(s) - sY(0) - Y'(0)$$

Example 8 Use the Laplace transform to solve the IVP:

$$Y' + 3Y = e^{2t}, \quad Y(0) = -1$$

• Let $\mathcal{L}[Y(t)] = Y(s)$. Apply the Laplace to the DE

$$\mathcal{L}[Y' + 3Y] = \mathcal{L}[e^{2t}]$$

$$\mathcal{L}[Y'] + 3\mathcal{L}[Y] = \frac{1}{s-2}$$

$$sY(s) - Y(0) + 3Y(s) = \frac{1}{s-2}$$

$$sY(s) + 1 + 3Y(s) = \frac{1}{s-2}$$

$$(s+3)Y(s) = \frac{1}{s-2} - 1$$

$$Y(s) = \frac{1}{(s+3)(s-2)} - \frac{1}{s+3}$$

$$\frac{1}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}$$

$$1 = (s+3)A + B(s-2) \Rightarrow A = \frac{1}{5}, \quad B = -\frac{1}{5} \Rightarrow \text{Cont}$$

$$Y(t) = \mathcal{L}^{-1}[Y(s)]$$

$$= \mathcal{L}^{-1}\left[\frac{1}{(s+3)(s-2)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s+3}\right]$$

$$Y(t) = \mathcal{L}^{-1}\left[\frac{1/5}{s-2} + \frac{-1/5}{s+3}\right] = \frac{1}{5}e^{2t} - \frac{1}{5}e^{-3t}$$

$$Y(t) = \frac{1}{5}e^{2t} - \frac{1}{5}e^{-3t}$$

$$Y(t) = \frac{1}{5}e^{2t} - \frac{6}{5}e^{-3t}$$

Or $Y' + 3Y = e^{2t}$, $Y(0) = -1$

$$\mu = e^{\int 3 dt} = e^{3t}$$

$$Y(t) = \frac{\int e^{3t} \cdot e^{2t} dt + C}{e^{3t}} = \frac{\frac{e^{5t}}{5} + C}{e^{3t}} = \frac{1}{5}e^{2t} - \frac{6}{5}e^{-3t}$$

$$Y(0) = -1 \Rightarrow C + \frac{1}{5} = -1 \Rightarrow C = -\frac{6}{5}$$

Example 3 Use Laplace transform to solve the IVP-

$$Y'' - Y' - 2Y = 0, \quad Y(0) = 1, \quad Y'(0) = 0$$

$$\mathcal{L}[Y'' - Y' - 2Y] = \mathcal{L}[0]$$

$$\mathcal{L}[Y''] - \mathcal{L}[Y'] - 2\mathcal{L}[Y] = 0$$

$$s^2 Y(s) - sY(0) - Y'(0) - s(Y(s) - Y(0)) - 2Y(s) = 0$$

$$s^2 Y(s) - s - sY(s) + 1 - 2Y(s) = 0$$

$$(s^2 - s - 2)Y(s) = s - 1$$

$$Y(s) = \frac{s-1}{s^2 - s - 2}$$

$$Y(t) = \mathcal{L}^{-1}[Y(s)] \Rightarrow \mathcal{L}^{-1}\left[\frac{s-1}{s^2 - s - 2}\right] = \mathcal{L}^{-1}\left[\frac{1/3}{s-2} + \frac{2/3}{s+1}\right]$$

$$= \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

Example 8 Use the laplace transform to solve 8

$$y'' - 6y' + 5y = 3e^{2t}$$

$$y(0) = 2, y'(0) = 3$$

$$L[y''] - 6L[y'] + 5L[y] = 3L[e^{2t}]$$

$$s^2 Y(s) - 2s - 3 - 6(sY(s) - 2) + 5Y(s) = \frac{3}{s-2}$$

$$(s^2 - 6s + 5)Y(s) = \frac{3}{s-2} + \frac{2s-9}{1}$$

$$(s-1)(s-5)Y(s) = \frac{2s^2 - 13s + 21}{s-2}$$

$$Y(s) = \frac{2s^2 - 13s + 21}{(s-1)(s-5)(s-2)} \Rightarrow \text{Partial Fraction Decomposition}$$

$$Y(s) = \int^{-1} [Y(s)] = \int^{-1} \left[\frac{5/2}{s-1} - \frac{1}{s-2} + \frac{1/2}{s-5} \right]$$

$$= \frac{5}{2} e^t - e^{2t} + \frac{1}{2} e^{5t}$$

Example 8 Solve the IVP 8

$$y' + 3y = \begin{cases} 0 & 0 \leq t \leq 2 \\ e^t + 1 & t \geq 2 \end{cases}$$

$$(y(0) = 1)$$

$$\text{the DE} \Rightarrow y' + 3y = (e^t + 1)u(t-2)$$

$$L[y'] + 3L[y] = L[(e^t + 1)u(t-2)]$$

$$sY(s) + 3Y(s) - 1 = e^{-2s} L[e^{t+2} + 1]$$

$$(s+3)Y(s) - 1 = e^{-2s} L[e^{-t-2} + 1]$$

$$(s+3)Y(s) = e^{-2s} \left(\frac{e^{-2}}{s+1} + \frac{1}{s} \right) + 1$$

$$Y(s) = e^{-2s} \left(\frac{e^{-2}}{(s+1)(s+3)} + \frac{1}{s(s+3)} \right) + \frac{1}{s+3}$$



$$Y(t) = \mathcal{L}^{-1}[Y(s)]$$

$$= \frac{e^{-2}}{2} u(t-2) \left(e^{-(t-2)} - e^{-3(t-2)} \right) + u(t-2) \left(\frac{1}{3} - \frac{1}{3} e^{-3(t-2)} \right) + e^{-3t}$$

$$\frac{1}{(s+1)(s+3)} = \frac{A}{(s+1)} + \frac{B}{(s+3)}$$

$$1 = A(s+3) + B(s+1)$$

$$A \rightarrow 1/2$$

$$B \rightarrow -1/2$$

$$\mathcal{L}^{-1} \left[\frac{1}{(s+1)(s+3)} \right] = \frac{1}{2} e^{-t} - \frac{1}{2} e^{-3t}$$

$$\frac{1}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}$$

$$A = 1/3, B = -1/3$$

$$\mathcal{L}^{-1} \left[\frac{1}{s(s+3)} \right] = \frac{1}{3} - \frac{1}{3} e^{-3t}$$

Example 3 The Current $I(t)$ in an LC Series Circuit is given by .

$$I''(t) + 4I(t) = g(t)$$

$$I(0) = 1, I'(0) = 3$$

$$g(t) = \begin{cases} 3 \sin t & 0 \leq t \leq 2\pi \\ 0 & t > 2\pi \end{cases} = 3 \sin t - 3 \sin t u(t-2\pi)$$

$$\text{Now } \mathcal{L}[I(t)] = F(s)$$

$$I'' + 4I = 3 \sin t - 3 \sin t u(t-2\pi)$$

$$\mathcal{L}[I''] + 4\mathcal{L}[I] = \frac{3}{s^2+1} - 3e^{-2\pi s} \mathcal{L}[\sin(t+2\pi)]$$

$$s^2 F(s) - sI(0) - I'(0) + 4F(s) = \frac{3}{s^2+1} - \frac{3e^{-2\pi s}}{s^2+1}$$

$$F(s) = \frac{3}{(s^2+1)(s^2+4)} - \frac{3e^{-2\pi s}}{(s^2+1)(s^2+4)} + \frac{s+3}{s^2+4}$$

$$\frac{3}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$3 = (As+B)(s^2+4) + (Cs+D)(s^2+1)$$

$$s=0 \Rightarrow 3 = 4B + D$$

$$s=1 \Rightarrow 3 = 5(A+B) + 2(C+D)$$

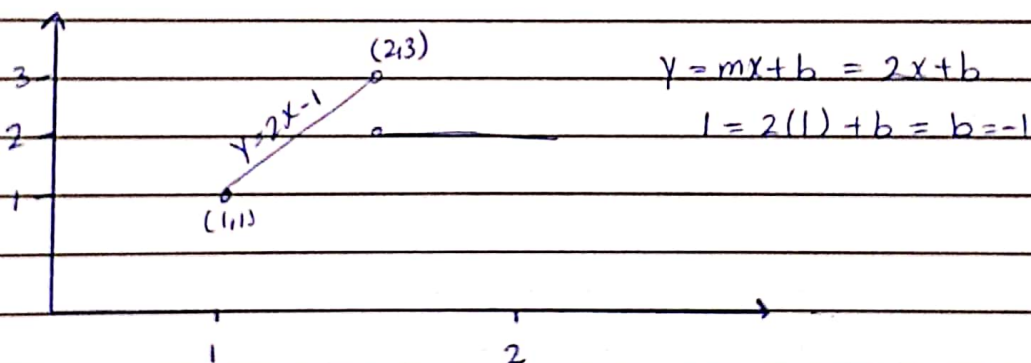
$$s=-1 \Rightarrow 3 = -5A - 5B - 2C + 2D$$

$$s=2 \Rightarrow 3 = 8(2A+B) + 5(2C+D)$$

$$(A=0, B=1, C=0, D=-1)$$

$$\frac{3}{(s^2+1)(s^2+4)} = \frac{1}{s^2+1} - \frac{1}{s^2+4}$$

$$(I(t) = \sin t - \frac{1}{2} \sin 2t - U(t-2\pi)(\sin t - 2\pi) - \frac{1}{2} \sin t + \cos t + \frac{3}{2} \sin t)$$



$$f(t) = \begin{cases} 2t-1 & 1 < t < 2 \\ 0 & t \geq 2 \end{cases} = 2t-1 + (3-2t)U(t-2)$$

$$L[f(t)] = \frac{2}{s^2} - \frac{1}{s} + e^{-2s} \left[3 - 2(t+2) \right]$$

$$= \frac{2}{s^2} - \frac{1}{s} + e^{-2s} \left(-\frac{1}{s} - \frac{2}{s^2} \right)$$

Example 2: $I(t)$ in RLC Series

$$I''(t) + 4I(t) = g(t)$$

$$I(0) = 1, I'(0) = 3$$

$$g(t) = \begin{cases} 3 \sin t, & 0 \leq t \leq 2\pi \\ 0, & 2\pi < t \end{cases}$$

20

$$y'' + 4y = 3\sin t - 3\sin t u(t-2\pi)$$

$$s^2 Y(s) - s - 3 + 4 Y(s) = \frac{3}{s^2+1} - \frac{3 e^{-2\pi s}}{s^2+1}$$

$$Y(s) = \frac{3}{(s^2+1)(s^2+4)} - \frac{3 e^{-2\pi s}}{(s^2+1)(s^2+4)} + \frac{s+3}{s^2+4}$$

$$\mathcal{L}^{-1} \left[\frac{e^{-2s} - 3 e^{-4s}}{s+3} \right]$$

$$= \mathcal{L}^{-1} \left[\frac{e^{-2s}}{s+3} \right] - 3 \mathcal{L}^{-1} \left[\frac{e^{-4s}}{s+3} \right]$$

$$= u(t-2) e^{-3(t-2)} - 3 u(t-4) e^{-3(t-4)}$$