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دفتر :

معادلات تفاضلية عادية

Ordinary Differential Equations

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(CH (1))

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معادلة Equation $\rightarrow 2x + 5 = 0 \rightarrow$ المعادلة التفاضلية $\rightarrow$ المعادلة التفاضلية

* Differential Equation (D.E)

$\rightarrow$ is an equation that contains at least one derivative.

$$* \dot{y} + 2x = e^x \rightarrow \text{D.E}$$

$$* y'' + 3xy' = 3e^{2x} \rightarrow \text{D.E}$$

$\dot{y} = \frac{dy}{dx}$
dependend $\rightarrow x \rightarrow$ independend.

Note $y^{(n)} \neq y^n \rightarrow$ Power
derivative $\rightarrow$

* Classification of D.E

$\rightarrow$ ordinary D.E $\rightarrow$ O.D.E $\rightarrow$ one independent Var
 $\rightarrow$ Partial D.E $\rightarrow$ P.D.E $\rightarrow$ that are at least two indep

$$* \frac{dy}{dx} = yx, \quad \frac{du}{dx} = Ux \quad \text{معادلة تفاضلية}$$

$$\text{Ex } \frac{d^2u}{dx^2} - \frac{du}{dt} = 0 \rightarrow \text{P.D.E} \rightarrow \text{Suppose } u \text{ is indep}$$

$\rightarrow u(x, t)$
dependend $\rightarrow$ independend

* order of D.E is the highest derivatives.

Ex Find the order of D.E

I $(\sin x) y = x^2 y'' - y' \rightarrow$ Third order

II $(y'')^3 - ty' = t^2 \rightarrow \text{order} = 2$

III $y^{(12)} [y + y^{(1)}]'' = y'^8$
 $\hookrightarrow y^{(12)} [y'' + y^{(3)}] = y'^8 \leadsto \text{order} = 13$

* Linear D.E $\rightarrow$ (dep. var) کے لئے

→ D.E of order n is called linear if it has form:

$$a_n y^n + a_{n-1} y^{n-1} + \dots + a_1 y' + a_0 y = f(x)$$

* شذوذ الخط (Linear)

① اعلى قوة Δ = Δ (y او مشتقاته y)

ع) مجموعہ خواجہ y_1 او y_2 (ف) مال کانسہ $y = dep$

is the following D.E linear or non linear?

III $y' - 5 \cos x = 0 \rightarrow$ linear

⑧ $(y)^2 - 5y = 3 \tan x \rightarrow$ non linear

3) $x''' + 3tx' + x^2 = t^3 \rightarrow$ not linear

↳ 11) Power is $\propto R$ & dep. on

المثال ① $y''' = axy - e^y$

- Third order
- O.D.E
- Non linear because (e^y)

② $xx' - tX'' = t^3$

- Second order
- O.D.E
- Non linear.

* Solution. D.E $\Rightarrow$ is a function that ~~satisfy~~ satisfy the
D.E $\Rightarrow y = f(x) \Rightarrow$ الدالة التي تحقق المعادلة

Defn Sol 8

$y = f(x)$ is function Sol to D.E if y satisfies P.E

two type of Sol :-

① Explicit Sol $\rightarrow y = f(x)$ (لا يكون البداية - بطرف)

② Implicit Solution: $\rightarrow$ خفي

Ex $y - \ln y = e^{xy}$

EX) Let $y'' - \frac{2}{x^2}y = 0$, $x \neq 0$. Show that $h(x) = x^2 - x^{-1}$ is Sol.

$(h'' - \frac{2}{x^2}h = 0)$ المطلوب

* $h' = 2x + x^{-2}$ } * $2 - 2x^{-3} - \frac{2}{x^2}(x^2 - x^{-1}) \stackrel{?}{=} 0$

* $h'' = 2 - 2x^{-3}$

$\rightarrow 2 - \frac{2}{x^3} - 2 + \frac{2}{x^3} = \text{zero}$

* Show that $y - \ln y = x^2 + 1$ is Sol to D.E $(y-1)y' = 2xy$

$\rightarrow$ Use I.D $\rightarrow y' - \frac{1}{y}y' = 2x$

$\rightarrow y y' - y' = 2xy$ (نضرب المعادلة بـ y)

$\rightarrow y(y-1) = 2xy$ *

Same form of D.E.

* Show that $y'' + (x^2+1)y - 3x^3 - 6 = 0$ is Sol for

$2xy^2 - 9x^2y + (2y^2 + x^2y + y)y' = 0$.

(H.W)

حل الواجب %

$$y^2 + (x^2 + 1)y - 3x^3 - 6 = 0$$

$$\rightarrow y^2 + yx^2 + y - 3x^3 - 6 = 0$$

$$\rightarrow 2yy' + 2yx + x^2y' + y' - 9x^2 = 0$$

نضرب بـ y $\rightarrow 2y^2y' + 2y^2x + y^2y' - 9x^2y + yy' = 0$

$$\rightarrow y'(2y^2 + y + yx^2) + 2y^2x - 9x^2y = 0 \quad *$$

* If $y = e^{-5x}$ is Sol for $y'' + xy' + P(x)y = 0$ Find $P(x)$:

$$\begin{aligned} \rightarrow y &= e^{-5x} \\ y' &= -5e^{-5x} \\ y'' &= 25e^{-5x} \end{aligned}$$

$$\rightarrow 25e^{-5x} + x(-5e^{-5x}) + P(x)e^{-5x} = 0$$

$$\downarrow e^{-5x} (25 - 5x + P(x)) = 0$$

$25 \neq 0$

$$\rightarrow P(x) - 5x + 25 = 0$$

$$\rightarrow P(x) = 5x - 25 \quad \times$$

* If $y = x^m$ is Sol for $x^2 y'' + 6xy' + 4y = 0, x > 0$
Find m !

$$\begin{aligned} \rightarrow y &= x^m \\ y' &= m x^{(m-1)} \\ y'' &= m(m-1) x^{(m-2)} \end{aligned}$$

$$\rightarrow x^2 m(m-1) x^{(m-2)} + 6x m x^{(m-1)} + 4x^m = 0$$

$$\rightarrow m(m-1) x^m + 6m x^m + 4x^m = 0$$

$$\rightarrow x^m (m^2 - m + 6m + 4) = 0$$

$$\rightarrow x^m (m^2 + 5m + 4) = 0$$

$x > 0$

$$\rightarrow (m+4)(m+1) = 0$$

$0 \neq 0$

$$\downarrow \quad \downarrow$$

$$m = -4 \quad m = -1 \quad \times$$

Initial value problem (I.V.P)

$$\downarrow \text{I.V.P} = \text{D.E} + \text{I.C}$$

Defen IVP of order (n) has the form

$$y^{(n)} = F(x, y, \dots, y^{(n-1)})$$

With Initial Condition (I.C)

$$\left[\begin{array}{l} y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \dots, \quad y^{(n-1)}(x_0) = y_{n-1} \end{array} \right]$$

* order = n ^{مرتبة}

(n-1) = أعلى مشتقة *

[n] = إذا الأولي *

* عند الشروط يجب أن تكون

Where $y_0, y_1, \dots, y_{n-1}$ are constant

x_0 is fixed ^{ثابت}

مع التثبيت I.C

EX ① $y' + 3y = x^2$
D.E

With

$$\frac{y_0 = 4}{\text{I.C}}$$

$$\frac{x_0 = 1}{y_0 = 4}$$

② $x^2 y'' + 4xy' + y = 0$, $y(0) = 4$, $y'(0) = 4$

هذا x_0 ليس

Solution should be IC , IC كل مسألة تأتي مع IC

* the IC use to find the constant integral soln

(note) **EX** if $y = c_1 e^{-x} + c_2 e^{2x}$ is sol to (IVP)

$$y'' - y - 2y = 0, \quad y(0) = 2, \quad y'(0) = -3$$

Find c_1, c_2 ?!

* كيف تأتي IVP
بحوضه الحل في
المعادلة وتبدأ بمشتقة
الشروط التي نحتاج

* $y(0) = 2$

$\rightarrow 2 = C_1 + C_2 \quad \dots \textcircled{1}$

* $y' = -C_1 e^{-x} + 2C_2 e^{2x} \quad \rightsquigarrow \quad \text{use } y'(0) = -3$

$\rightarrow -3 = -C_1 + 2C_2 \quad \dots \textcircled{2}$

$\textcircled{1} + \textcircled{2} \Rightarrow -1 = 3C_2 \rightsquigarrow C_2 = \boxed{-\frac{1}{3}}$

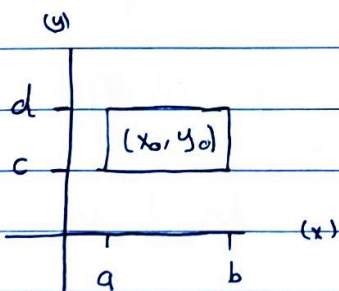
$C_1 = 2 + \frac{1}{3} = \boxed{\frac{7}{3}}$

* Exist and unique

consider I.V.P. $y' = f(x, y)$, $f(x_0) = y_0$

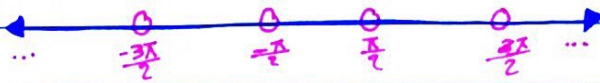
if $f(x, y)$ and $\frac{df}{dy}$ are cont on $a < x < b$ and $c < y < d$ contains (x_0, y_0) then IVP has unique sol

where $\frac{df}{dy}$ is partial derivative of f with respect to y



[2] $y' + (\tan x)y = \sin x$, $y(x) = 0$

↳ $y' = \sin x - (\tan x)y \Rightarrow f(x,y)$ cont on $\mathbb{R} \setminus \{x = \pm(2n+1)\frac{\pi}{2} \mid n \geq 0\}$
Interval of cont for $f(x,y)$



* $\frac{df}{dy} = -\tan x$ is ~~not~~ cont on $\mathbb{R} \setminus \{x = \pm(2n+1)\frac{\pi}{2} \mid n \geq 0\}$, $n \geq 0$

* As $x_0 = \pi \in (\frac{\pi}{2}, \frac{3\pi}{2})$

↳ Then $(\frac{\pi}{2}, \frac{3\pi}{2})$ is largest interval for unique Sol

[3] $y' + \frac{y}{\ln(\ln x - 1)} = \frac{x}{x-2}$, $y(\frac{e}{2}) = 0$ (H.w)

CH(2) "First order D.E"

(I) Separable D.E

↳ if $y' = f(x,y)$ can be written as

↳ $f(x) dx = g(y) dy$

where $f(x)$ is fn of x -only

$g(y)$ is fn of y -only

Then, D.E is called ~~not~~ Separable D.E.

Solve
(Ex) $y' = \frac{x^2+1}{y^2-1} \Rightarrow \frac{dy}{dx} = \frac{x^2+1}{y^2-1} \Rightarrow (x^2+1) dx = (y^2-1) dy$ (Sep. eqn)

Method Find Sol for Sep. eqn by Integrate both sides.

$C + \int f(x) dx = \int g(y) dy$

Sol: $\int (x^2+1) dx = \int (y^2-1) dy \Rightarrow \frac{x^3}{3} + x + C = \frac{y^3}{3} - y$

∫ f(x) dx = I.C

∫ g(y) dy = I.C

Ex) Solve $y' = 1+y^2 (\tan x)$, $y(0) = \sqrt{3}$

$\hookrightarrow \frac{dy}{dx} = (1+y^2) \tan x \rightarrow \frac{dy}{1+y^2} = \tan x \, dx$

Sep. eqn $\int \frac{dy}{1+y^2} = \int \tan x \, dx$

$\hookrightarrow \tan^{-1} y = -\ln |\cos x| + C$

I.C. at $x=0$ is

$(x_0 = 0)$
 $(y_0 = \sqrt{3})$

$\hookrightarrow \tan^{-1} \sqrt{3} = -\ln |\cos 0| + C$

$\hookrightarrow \frac{\pi}{3} = C$

$\hookrightarrow$ General Sol

$\hookrightarrow \tan^{-1} y = -\ln |\cos x| + \frac{\pi}{3}$

Ex) Solve $y' = \frac{y \cos(x)}{1+3y^2}$, $y(0) = 1$ (H.W)

* Separable eqn. $y' = \frac{dy}{dx}$

$$\int f(x) dx = \int g(y) dy$$

$+C$

Ex) Solve: $y' = y^2 - 4$

$$\rightarrow y' = \frac{dy}{dx} = y^2 - 4 \rightarrow \frac{1}{y^2 - 4} dy = dx \quad (\text{separable})$$

$$\rightarrow \int dx = \int \frac{1}{y^2 - 4} dy \Rightarrow$$

$X + C =$ $\frac{1}{y^2 - 4}$ partial fraction

$$\rightarrow \frac{1}{y^2 - 4} = \frac{1}{(y-2)(y+2)} = \frac{A}{y-2} + \frac{B}{y+2} \quad \text{Covers up}$$

$$\rightarrow A(y+2) + B(y-2) = 1$$

$$y=2 \rightarrow 4A = 1 \rightarrow \boxed{A = \frac{1}{4}}, \quad y=-2 \rightarrow -4B = 1 \rightarrow \boxed{B = -\frac{1}{4}}$$

$$\rightarrow X + C = \int \left(\frac{\frac{1}{4}}{y-2} + \frac{-\frac{1}{4}}{y+2} \right) dy$$

$$\rightarrow X + C = \frac{1}{4} \ln|y-2| - \frac{1}{4} \ln|y+2|$$

$$\rightarrow X + C = \frac{1}{4} \ln \left| \frac{y-2}{y+2} \right|$$

* III Linear Equation

dep var $\rightarrow$ dep var

Form of D.E linear in y

$$* y' + P(x)y = Q(x)$$

$\rightarrow$ where $Q(x)$ and $P(x)$ function of x -only, (1) p & y solve

Ex) $y' + 2y = e^x$ ✓ (Linear)

② $y' + x e^y = \sin x$ ✗ (Not Linear)

* Linear in x

$$\rightarrow x' + P(y)x = Q(y)$$

Method 8 $y' + P(x)y = Q(x)$

① Find integrating Factor , $M(x) = e^{\int P(x) dx}$

$$M'(x) = \underbrace{e^{\int P(x) dx}}_{M(x)} \cdot P(x) \rightsquigarrow P(x) = \frac{M'(x)}{M(x)}$$

② Multiply I.F by D.E 8 $\underbrace{M(x)} y' + \underbrace{M(x) P(x)}_{M'(x)} y = \underbrace{M(x)} Q(x)$

$$\hookrightarrow M(x) y' + M'(x) y = M(x) Q(x)$$

$$(M(x) y)' = M Q \quad \text{Si افاد الـ طرفين}$$

$$\int (M(x) y)' dx = \int M \cdot Q dx$$

$$M(x) y(x) = \int M \cdot Q dx + \underline{\underline{C}}$$

ثابت التكامل عند الطرف الأيمن

$$\text{Sol} \hookrightarrow y(x) = \frac{1}{M(x)} \int M \cdot Q dx + C$$

Ex Solve $\rightarrow xy' + y = \sin x, \quad x > 0$

$$y' + \frac{1}{x} y = \frac{\sin x}{x}$$

* Linear in y with $P(x) = \frac{1}{x}, \quad Q(x) = \frac{\sin x}{x}$

$$① M(x) = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

② Sol:

$$\hookrightarrow y = \frac{1}{x} \left[\int x \cdot \frac{\sin x}{x} dx + C \right]$$

$$\hookrightarrow \frac{1}{x} \left[\int \sin x dx + C \right] \rightsquigarrow y(x) = \frac{1}{x} [-\cos x + C]$$

Ex.) Solve: $ty' - 2y = 5t^2$, $y(1) = 2$, $t > 0$

1st step $y' - \frac{2}{t}y = 5t$

↓ linear in y with $P(t) = \frac{-2}{t}$, $Q(t) = 5t$

① $M(t) = e^{\int \frac{-2}{t} dt} = e^{-2 \ln t} = e^{\ln t^{-2}} = t^{-2} = \frac{1}{t^2}$

② Sol. $y(t) = t^2 \left[\int \frac{1}{t^2} \cdot 5t dt + C \right] = t^2 \left[\int 5t^{-1} dt + C \right]$

$y(t) = t^2 [5 \ln t + C]$

* to find C use I.C. $y(1) = 2$

↓ $2 = 1 [5 \ln 1 + C]$ $\rightarrow C = 2$

↓
* $y(t) = t^2 [5 \ln t + 2]$

Ex.) If $M(x) = 3x^2$ is an I.F. for

$2y' + x \sin(x) y = 3x^2 - 4$

Find $S(x)$!

→ $y' + x \sin(x) y = \frac{3}{2}x^2 - 2$

→ $P(x) = \frac{M'}{M} = \frac{6x}{3x^2} = \frac{2}{x} = \frac{x \sin(x)}{2} \rightarrow S(x) = \frac{4}{x^2}$

1] Separable . eqn

$$\int f(x) dx + C = \int g(y) dy$$

2] Linear D.E

$$\begin{cases} \text{in } y \rightarrow \dot{y} + P(x)y = Q(x) \\ \text{in } x \rightarrow \dot{x} + P(y)x = Q(y) \end{cases}$$

$$* M(x) = e^{\int P(x) dx} \rightarrow y = \frac{1}{M(x)} \left[\int M(x) \cdot Q(x) dx + C \right]$$

$$* M(y) = e^{\int P(y) dy} \rightarrow x = \frac{1}{M(y)} \left[\int M(y) \cdot Q(y) dy + C \right]$$

(Ex) Solve :- $(x+4y^2) dy + 2y dx = 0$

نقسم على dx * $(x+4y^2) \frac{dy}{dx} + 2y = 0$

نقسم على y * $\frac{(x+4y^2)}{x+4y^2} y' + \frac{2y}{x+4y^2} = 0$

نقلب المعادلة * $\frac{dy}{dx} = \frac{x+4y^2}{-2y}$

نوزع المقام * $x' = x \left(\frac{-1}{2y} \right) - 2y \rightarrow \bar{x} + \frac{1}{2y} x = -2y$

* Linear in x with $P(y) = \frac{1}{2y}$, $Q(y) = -2y$

$$\rightarrow M(y) = e^{\int \frac{1}{2y} dy} \rightarrow e^{\frac{1}{2} \ln y} = \sqrt{y}$$

$$\rightarrow x = \frac{1}{\sqrt{y}} \left[\int \sqrt{y} \cdot (-2y) dy + C \right] \rightarrow \frac{1}{\sqrt{y}} \left[-2 \int y^{\frac{3}{2}} dy + C \right]$$

$$x = \frac{1}{\sqrt{y}} \left[-2 \cdot \frac{2}{5} y^{\frac{5}{2}} + C \right]$$

$$x = \frac{1}{\sqrt{y}} \left[-\frac{4}{5} y^{\frac{5}{2}} + C \right]$$

3 Bernoulli Eqn

D.E $\hookrightarrow y' + P(x)y = Q(x)y^n, \quad n \neq 0, 1$

* If $n=0 \rightarrow$ D.E is linear.

* If $n=1 \rightarrow$ D.E is separable.

Ex $y' + 2x y = 1 y^{\frac{3}{2}}$

Method: let $(V = y^{1-n})$

Transform D.E (Bernoulli) to linear in V.

$\hookrightarrow y' + P(x)y = Q(x)y^n$

$\hookrightarrow y' y^{1-n} + P(x) y^{1-n} = Q(x)$ (y^n divide)

$\hookrightarrow \frac{V'}{(1-n)y^n} + P(x)V = Q(x)$ (multiply)

$\hookrightarrow V' + (1-n)P(x)V = (1-n)Q(x)$ (multiply)

$\hookrightarrow$ Linear in V

* Find the sol in V

* Use $V = y^{1-n}$ To find sol in y

let $V = y^{1-n}$

$\hookrightarrow V' = (1-n)y^{-n} y'$

$\hookrightarrow y' = \frac{V'}{y^n(1-n)}$

Ex) Solve $y' - 5y = -\frac{5}{2} x y^3$

$\hookrightarrow$ Bernoulli eqn with $P(x) = -5, \quad Q(x) = -\frac{5}{2}x$ if $n = 3 \neq 0, 1$

* let $v = y^{1-3} = y^{-2}$

* Linear in V: $V' + -2 = -5 * V = -2 + -\frac{5}{2}x$

$\hookrightarrow V' + 10V = 5x$

I.F $\hookrightarrow M(x) = e^{\int 10} = e^{10x}$

$\hookrightarrow v(x) = \frac{1}{e^{10x}} \left[\int \frac{e^{10x}}{e^{10x}} * 5x dx + C \right]$

$\hookrightarrow$ by parts

* $v(x) = e^{-10x} \left[\frac{1}{2} x e^{10x} - \frac{1}{20} e^{10x} + C \right]$

$\hookrightarrow v(x) = \frac{1}{2} x - \frac{1}{20} + e^{-10x} C$

Sol in y (Use $v = y^{-2}$) $\hookrightarrow y^{-2} = \frac{1}{2}x - \frac{1}{20} + \frac{C}{e^{10x}}$

Ex) Solve $(x-2)y' - 5(x-2)^2 y^{\frac{1}{2}} = y$, $y(3) = 4$

$(x-2)$ $\downarrow$ $y' - 5(x-2) y^{\frac{1}{2}} = \frac{y}{(x-2)}$

$\downarrow$ $y' - \frac{1}{(x-2)} y = 5(x-2) y^{\frac{1}{2}}$

* Bernoulli with $P(x) = -\frac{1}{(x-2)}$, $Q(x) = 5(x-2)$, $n = \frac{1}{2} \neq 0, 1$

let $v = y^{\frac{1}{2}} = y^{\frac{1}{2}}$

linear in v :-

$\downarrow$ $v' + (1-n) P(x) v = (1-n) Q(x)$

$\downarrow$ $v' + \frac{1}{2} \cdot \frac{-1}{x-2} v = \frac{1}{2} \cdot 5(x-2)$

$\downarrow$ $v' - \frac{1}{2(x-2)} v = \frac{5}{2} (x-2)$

$\downarrow$ $\mu(x) = e^{\int \frac{1}{2(x-2)} dx} = e^{\frac{1}{2} \ln|x-2|} = (x-2)^{\frac{1}{2}}$

Sol: $\downarrow$ $v(x) = (x-2)^{\frac{1}{2}} \left[\int (x-2)^{\frac{1}{2}} \cdot \frac{5}{2} (x-2) dx + C \right]$

$\downarrow$ $v(x) = \sqrt{x-2} \left[\int (x-2)^{\frac{3}{2}} \cdot \frac{5}{2} dx + C \right]$

$\downarrow$ $v = \sqrt{x-2} \left[\frac{5}{2} \cdot \frac{2}{5} (x-2)^{\frac{5}{2}} + C \right]$

$\downarrow$ $v = \sqrt{x-2} \left[\frac{5}{2} (x-2)^{\frac{5}{2}} + C \right]$

* Use $v = y^{\frac{1}{2}}$

$\downarrow$ $y^{\frac{1}{2}} = \sqrt{x-2} \left[\frac{5}{2} (x-2)^{\frac{5}{2}} + C \right]$

Use I.C to find C [$x=3, y=4$]

$2 = \frac{5}{2} + C$

$\downarrow$ $y^{\frac{1}{2}} = \sqrt{x-2} \left[\frac{5}{2} (x-2)^{\frac{5}{2}} + \frac{1}{3} \right]$

$\downarrow$ $C = 2 - \frac{5}{2} = \frac{1}{2}$

* H.W) $x(x+2)y' = -4(1+x)y + 2(1+2x^2)\sqrt{y}$, $y > 0$

Ex) which of the following D.E is exact :-

① $(\cos x \cos y + 2x) dx - (\sin x \sin y + 2y) dy = 0$

* $M_y = -\cos x \sin y$, $N_x = -\cos x \sin y$

As $M_y = N_x = -\cos x \sin y \Rightarrow$ D.E is exact.

② $(2xy + x) = (x^2 - 1)y' \Rightarrow (2xy + x) - (x^2 - 1)y' = 0$

Form (1) $\rightarrow$

$M_y = 2x$, $N_x = -2x$

As $M_y \neq N_x \Rightarrow$ Not Exact.

Ex) Find the Value of k such that the D.E is Exact

$(6xy^3 + \cos y) dx + (kx^2y^2 - x \sin y) dy = 0$

$\hookrightarrow M_y = N_x$ as DE is exact

$\hookrightarrow 18xy^2 - \sin y = kx^2y^2 - \sin y$

$\hookrightarrow 18 = 2k \Rightarrow k = 9$

Ex) Determine $M(x,y)$ such that the D.E

$M(x,y) dx + (xe^{xy} + \frac{1}{x}) dy = 0$ is Exact.

As D.E is exact $\Rightarrow M_y = N_x$

$\hookrightarrow N_x = xe^{xy} + \frac{1}{x} \Rightarrow N_x = xye^{xy} + e^{xy} - \frac{1}{x^2} = M_y$

$\hookrightarrow M(x,y) = \int M_y dy = \int (xye^{xy} + e^{xy} - \frac{1}{x^2}) dy$

$\hookrightarrow x \int ye^{xy} dy + \int e^{xy} dy - \int \frac{1}{x^2} dy$

by parts

$\hookrightarrow M = x \left(y \frac{e^{xy}}{x} - \frac{e^{xy}}{x^2} \right) + \frac{e^{xy}}{x} - \frac{1}{x^2} y + h(x)$

$u = y$ $dv = e^{xy} dy$

$du = dy \rightarrow v = \frac{e^{xy}}{x}$

$\hookrightarrow M(x,y) = ye^{xy} - \frac{e^{xy}}{x} + \frac{e^{xy}}{x} - \frac{y}{x^2} + h(x)$

$\hookrightarrow M(x,y) = ye^{xy} - \frac{y}{x^2} + h(x)$

* If $M(x,0) = 0$, find $h(x)$

$0 = e^0 - \frac{1}{x^2} + hx \Rightarrow hx = 0 - e^0 + \frac{1}{x^2} \quad \#$

How* Find $M(x,y)$ Such that D.E is Exact
 $(y \cos(xy) + e^x) dx + N(x,y) dy = 0$

Method: to solve exact eqn:

$$M(x,y) dx + N(x,y) dy = 0$$

$$\text{and } M_x = N_y$$

Steps & ~~let~~ let the sol has form $F(x,y) = C$

$$\text{[I]} \frac{dF}{dx} = M \quad \text{or} \quad \frac{dF}{dy} = N$$

$$\text{[2]} F(x,y) = \int M(x,y) dx + g(y)$$

$$\text{[3]} \text{ Use } \frac{dF}{dy} = N \text{ to Find } g(y)$$

$$\text{[4]} \text{ Integrate to find } g(y).$$

$$\text{Ex) Solve } (\cos x \sin x - xy^2) dx + y(1-x^2) dy = 0$$

$$\hookrightarrow M_y = -2y = N_x = y(-2x)$$

$$\text{*Sol: } F(x,y) = C$$

$$\text{[1]} \frac{dF}{dx} = M = \cos x \sin x - xy^2 \quad \frac{1}{2} \sin 2x$$

$$\text{[2]} F(x,y) = \int M dx = \int (\cos x \sin x - xy^2) dx + g(y)$$

$$\hookrightarrow -\frac{1}{4} \cos 2x - \frac{x^2 y^2}{2} + g(y)$$

$$\text{[3]} \left[\frac{dF}{dy} = N \right] \Rightarrow \left[0 - x^2 y + g'(y) = y - yx^2 \right]$$

$$\boxed{g'(y) = y} \leadsto \text{[4]} g(y) = \frac{y^2}{2}$$

General Sol.

$$F(x,y) = C \leadsto -\frac{1}{4} \cos(2x) - \frac{x^2 y^2}{2} + \frac{y^2}{2} = C$$

Method (2) :

$$\rightarrow F(x, y) = \int (y - yx^2) dy$$

$$F = \frac{y^2}{2} - \frac{y^2}{2} x^2 + h(x)$$

$$\frac{dF}{dx} = M \rightarrow 0 - y^2 x + h'(x) = \cos x \sin x - xy^2$$

$$h'(x) = \cos x \sin x$$

$$h'(x) = \frac{1}{2} \sin 2x$$

Integrate

$$h(x) = \frac{-1}{4} \cos 2x$$

24. Oct. 2023

Exact Equation:-

$$M(x,y) dx + N(x,y) dy = 0$$

$$\Rightarrow M(x,y) + N(x,y) y' = 0$$

$$\rightarrow \text{If } \frac{dM}{dy} = \frac{dN}{dx} \Rightarrow \text{is exact}$$

$$\text{Sol: } F(x,y) = C$$

$$F(x,y) = \int N dy + h(x)$$

$$\frac{dF}{dx} = M, \quad \frac{dF}{dy} = N$$

$$h'(x) = \dots$$

Ex Solve: $(1 + e^y + xe^y) dx + (xe^x + 2) dy = 0$

$$M_y = e^x + xe^x \quad \equiv \quad N_x = xe^x + 1e^x = 0$$

* As $M_y = N_x \Rightarrow$ D.E is exact.

* Sol: $F(x,y) = C$

$$* F(x,y) = \int (xe^x + 2) dy = (xe^x + 2)y + h(x)$$

$$* \frac{dF}{dx} = M \Rightarrow (xe^x + e^x)y + h'(x) = 1 + e^y + xe^y$$

$$* h'(x) = 1$$

$$* h(x) = x$$

* general Sol: $F(x,y) = C$

$$(xe^x + 2)y + x = C$$

Ex₂ Solve: $(2xy - \sec^2 x) dx + (x^2 + 2y) dy = 0$

$$M_y = 2x \quad \equiv \quad N_x = 2x$$

* As $M_y = N_x \Rightarrow$ D.E is exact.

$$* \text{Sol: } F(x,y) = C = \int (x^2 + 2y) dy = x^2 y + y^2 + h(x)$$

$$* \frac{dF}{dx} = M \Rightarrow 2xy + 0 + h'(x) = 2xy - \sec^2 x$$

$$* h'(x) = -\sec^2 x$$

$$* h(x) = -\tan x$$

* General Sol: $F(x,y) = C$

$$\hookrightarrow x^2 y + y^2 - \tan x = C$$

Sol:

الحل بالطريقة الأخرى 8

$$* F(x,y) = \int 2xy - \sec^2 x \, dx$$

$$= yx^2 - \tan x + g(y).$$

$$* \frac{dF}{dy} = N \Rightarrow x^2 - 0 + g'(y) = x^2 + 2y$$

$$* g'(y) = +2y$$

$$* g(y) = y^2$$

$$* F(x,y) = C \Rightarrow yx^2 - \tan x + y^2 = C$$

general Sol $\rightarrow$

* Find $N(x,y)$ such that D.E is exact

$$\int (y \cos(x,y) + e^x) \, dx + N(x,y) \, dy = 0$$

$$\int x^2 y^3 \, dx + (N - 2x) \, dy = 0$$

$$\text{with } N(1,y) = 6$$

$$\# \text{ Let } M(x,y) \, dx + N(x,y) \, dy = 0$$

$$\text{But } \frac{dy}{dx} \neq \frac{dN}{dx} \Rightarrow (\text{Not exact})$$

$$\int_{M^*} (M(x,y) M) \, dx + \int_{N^*} (M(x,y) N) \, dy = 0$$

Trans Form to exact $\Rightarrow M(x,y)$ is called Special I.F

$$* \frac{dM^*}{dy} = \frac{dN^*}{dx} \Rightarrow \text{يعني ما يتحول الى exact زي قبل}$$

$$\text{Ex) If } (12 + 5xy) \, dy + (6xy' + 3x^2) \, dy = 0$$

$$\text{has I.F } M(x,y) = x^n y^m$$

Find n & m.

$$M_y = 5x \neq N_x = 6y' + 6x \Rightarrow \text{Not exact.}$$

$$* \text{multiply D.E by } M(x,y) = x^n y^m$$

عوض

Sol $\Rightarrow$ M^* N^*

$$(12x^n y^m + 5n^{n+1} y^{m+1}) dx + (6x^{n+1} y^{m+1} + 3x^{n+2} y^m) dy = 0$$

$$\bullet \frac{dM^*}{dy} = \frac{dN^*}{dx}$$

$$\Rightarrow \underline{12x^n (m y^{m-1})} + \underline{5(m+1) n^{n+1} y^m} = \underline{6(n+1) x^n y^{m+1}} + \underline{3(n+2) x^{n+1} y^m}$$

$$\begin{array}{l} (x^n y^{m-1}) \quad 12m = 6(n+1) \quad \underline{1} \\ (x^{n+1} y^m) \quad 5(m+1) = 3(n+2) \quad \underline{2} \end{array}$$

$$\div 6 \rightsquigarrow 2m = n+1 \Rightarrow n = 2m-1$$

(بقوة n في المعادلة الثانية)

$$* 5(m+1) = 3(2m-1+2)$$

$$* 5m+5 = 6m+3$$

$$* m = \underline{\underline{2}}$$

$$* n = 2*2-1 = \underline{\underline{3}}$$

Method to find Special I.F :-

considerd $\rightarrow M dx + N dy = 0$

$$\& M_y \neq N_x$$

II If $\frac{M_y - N_x}{N}$ depend only on x .

$$\rightarrow \text{special I.F } M(x) = e^{\int \frac{M_y - N_x}{N} dx} = \dots$$

III If $\frac{N_x - M_y}{M}$ depend only on y .

$$\rightarrow N(y) = e^{\int \frac{N_x - M_y}{M} dy} = \dots$$

$$* (y \cos(xy) + e^x) dx + N(x,y) dy = 0$$

exact

$$\rightarrow M_y = N_x$$

$$M_y = y * -\sin(xy) * x + \cos(xy) * 1 + 0 = N_x$$

$$N_x = -yx \sin(xy) + \cos(xy)$$

$$N = \int -yx \sin(xy) + \cos(xy) dx$$

$$\begin{array}{l} u = yx \\ du = y dx \end{array} \quad \begin{array}{l} dv = -\sin xy \\ v = \frac{\cos(xy)}{y} \end{array}$$

$$N = \frac{yx \cos(xy)}{y} - \int \frac{y \cos(xy)}{y} dx + \int \cos(xy) dx$$

$$N = x \cos(xy)$$

26-oct-2023

2.5 Not exact eqn:

$$Mdx + Ndy = 0$$

But $M_y \neq N_x \rightarrow$ "Not exact"

$M(x,y) \rightarrow$ Exact
 $\rightarrow$ special I.F

* to find I.F :

1] If $\frac{M_y - N_x}{N}$ only on $x \rightarrow$ I.F $\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}$

2] If $\frac{N_x - M_y}{M}$ only on $y \rightarrow$ I.F $\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}$

Ex If $M(y) = y$ is I.F that transform

$$Mdx + Ndy = 0 \text{ into exact.}$$

Find $N_x - M_y$!

I.F $M(y) = y$ function dep of y only

$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}$$

$$\downarrow y = e^{\int \frac{N_x - M_y}{M} dy} \rightarrow \ln y = \int \frac{N_x - M_y}{M} dy \quad \text{نسق الطرفين}$$

$$\rightarrow \frac{1}{y} = \frac{N_x - M_y}{M} \rightarrow N_x - M_y = -\frac{M}{y} \quad \times$$

Ex Find I.F that transform D.E to exact.

$$\downarrow (2x^2y + x^3)dx + (x^3 + xy^3)dy = 0$$

$$M_y = 2x^2 \neq N_x = 3x^2 + y^3 \rightarrow \text{Not exact}$$

$$\square \frac{M_y - N_x}{N} = \frac{2x^2 - 3x^2 - y^3}{x^3 + xy^3} = \frac{-x^2 - y^3}{x(x^2 + y^3)} = \frac{-1}{x} \text{ only in } x$$

$$\Rightarrow \text{I.F } \mu(x) = e^{\int \frac{-1}{x} dx} = x^{-1} = \frac{1}{x}$$

$$\underline{2} \quad (y^2 + 3xy^3) dx + (1 - xy) dy = 0$$

$$M_y = 2y + 9xy^2 \neq N_x = -y \quad (\text{Not exact})$$

$$\text{[1]} \quad \frac{M_y - N_x}{N} = \frac{2y + 9xy^2 + y}{1 - xy} = \frac{3y(1 + 3xy)}{1 - xy} \quad \text{test fail}$$

$$\text{[2]} \quad \frac{N_x - M_y}{M} = \frac{-y - 2y - 9xy^2}{y^2 + 3xy^3} = \frac{-3y(1 + 3xy)}{y^2(1 + 3xy)} = \frac{-3}{y} \quad \text{only in } y$$

$$M(y) = e^{\int \frac{-3}{y} dy} = e^{-3 \ln y} = y^{-3} = \frac{1}{y^3}$$

$$\Rightarrow \text{multiply D.E by } M(x) = \frac{1}{y^3}$$

إذن $\Rightarrow$ IF للـ eqn الـ جديد

$$\hookrightarrow \left(\frac{1}{y} + 3x\right) dx + \left(\frac{1}{y^3} - \frac{x}{y^2}\right) dy = 0$$

$$M_y = \frac{-1}{y^2} = N_x = \frac{-1}{y^2} \quad \underline{\text{exact eqn}}$$

Ex Find the general Sol :-

$$y dx + (2x - ye^y) dy = 0$$

$$\hookrightarrow M_y = 1 \neq N_x = 2 \quad \text{"Not exact"}$$

* Find I.F

$$\text{[1]} \quad \frac{M_y - N_x}{N} = \frac{1 - 2}{2x - ye^y} = \frac{-1}{2x - ye^y} \rightarrow (\text{Not only in } x)$$

$$\text{[2]} \quad \frac{N_x - M_y}{M} = \frac{2 - 1}{y} = \frac{1}{y} \rightarrow (\text{only in } y)$$

$$\Rightarrow \text{I.F} = M(y) = e^{\int \frac{1}{y} dy} = e^{\ln y} = \boxed{y}$$

multiply D.E by $M(y) = y$:-

$$* y^2 dx + (2yx - y^2 e^y) dy = 0$$

$$\hookrightarrow M_y = 2y = N_x = 2y - 0 \Rightarrow \text{exact eqn}$$

الـ Sol :-

* $F(x, y) = C$

↳ $F(x, y) = \int y^2 dx = y^2 x + g(y)$

↳ $\frac{dF}{dy} = N \Rightarrow 2yx + g'(y) = 2xy - y^2 e^y$

↳ $g'(y) = -y^2 e^y$ اجزاء

u	dv
y^2	e^y
$2y$	e^y
2	e^y
0	e^y

↳ $g(y) = -(y^2 e^y - 2y e^y + 2e^y)$
 $= -y^2 e^y + 2y e^y - 2e^y$

general sol:

↳ $F(x, y) = C$

↳ $y^2 x + 2y e^y - y^2 e^y - 2e^y = C$

* بطريقة حل اخره الحالة معينة خلايا بطريقة linear

↳ $y dx + (2x - y e^y) dy = 0 \quad \div dy$

↳ $y x' = -2x + y e^y \quad \div y$

↳ $x' + \frac{2}{y} x = e^y$
 $P(y) \quad Q(y)$

Q] Consider D.E $M dx + N dy = 0$ If

$(M_y)^2 - (N_x)^2 = \cos x (M_y + N_x) N$

Find I.F for D.E.

$\cos x > 0, M_y + N_x > 0$

Sol 8 $(M_y - N_x)(M_y + N_x) = \cos x (M_y + N_x) N$

$\div N$

$\frac{M_y - N_x}{N} = \cos x$

بتردد e و $\sin x$

$e^{\int \frac{M_y - N_x}{N} dy} = e^{\int \cos x dx} \Rightarrow \mu(x) = e^{\sin x}$

Q) If $\frac{M(x,y) - N(x,y)}{N(x,y)} = x$

Find the general Sol to D.E

* $\frac{M}{x} dx - \frac{N}{y} dy = 0$, $xy > 0$

$$\frac{M-N}{N} = x \Rightarrow \frac{M}{N} - 1 = x \Rightarrow \frac{M}{N} = x+1$$

بجواب المعادله
 $\frac{1}{N} \rightarrow \Rightarrow \frac{1}{x} * \frac{M}{N} dx - \frac{1}{y} dy = 0$

$$\Rightarrow \frac{x+1}{x} dx - \frac{1}{y} dy = 0 \Rightarrow \text{Sep}$$

2.6 Homog D.E

$$y' = G(ax+by+c)$$

$$y' = F\left(\frac{y}{x}\right) \text{ OR } y' = F\left(\frac{x}{y}\right)$$

$\hookrightarrow y' = 1 + \frac{y}{x} + e^y \Rightarrow \text{Not Homog}$

$$\approx y' = 1 + \frac{y}{x} + e^{\frac{x}{y}}$$

$$\hookrightarrow y' = 1 + \frac{y}{x} + e^{\left(\frac{x}{y}\right)^{-1}} \quad \checkmark$$

26 Homog. D.E

Defn: In any First order D.E $y' = f(x, y)$
can be written in the form:

$$y' = F\left(\frac{y}{x}\right) \text{ or } y' = F\left(\frac{x}{y}\right)$$

Ex: $y' = 1 + \frac{y}{x} + e^{\frac{y}{x}}$ $\leadsto$ Not Homog.

$$y' = 1 + \frac{y}{x} + e^{\frac{y}{x}} = 1 + \underbrace{\left(\frac{y}{x}\right)}_{F\left(\frac{y}{x}\right)} + e^{\frac{y}{x}}$$

* Note *

$$\hookrightarrow \frac{x}{y} = \left(\frac{y}{x}\right)^{-1} \text{ Call } \rightarrow \left(\frac{y}{x}\right) \text{ by using substitution}$$

$$y' + \ln x = \frac{2x - y}{x + 5y} + \ln y$$

$$\hookrightarrow y' = \frac{2x - y}{x + 5y} + (\ln y - \ln x) \rightarrow \ln \text{ is separable}$$

$$y' = \frac{2x - y}{x + 5y} + \ln \frac{y}{x}$$

$$y' = \frac{x(2 - \frac{y}{x})}{(1 + 5\frac{y}{x})} = \ln \frac{y}{x} \leadsto \text{Homog}$$

* Method: let $y' = F\left(\frac{y}{x}\right)$

Inside: $u = \frac{y}{x} \Rightarrow y = xu \leadsto y' = xu' + u$
in D.E: $xu' + u = F(u)$

$$x \frac{du}{dx} = F(u) - u$$

$$\frac{x}{dx} = \frac{F(u) - u}{du}$$

$$\int \frac{dx}{x} = \int \frac{du}{F(u) - u}$$

$$\ln|x| + C = \int \frac{du}{F(u) - u}$$

$u = \frac{y}{x}$ Transform
D.E into
separable

$$\frac{A}{B} - \frac{C}{D} = \frac{B}{A} - \frac{D}{C}$$

* General Sol in y : Use $u = \frac{y}{x} \Rightarrow y = xu$

Ex) Solve: $x^2 y' = (x^2 + xy + y^2)$, $x > 0$

لأننا نريد بالدرجة الأولى
 $x^2 dy = (x^2 + xy + y^2) dx \rightarrow x^2 dy = (x^2 + xy + y^2) dx$

لأننا نريد بالدرجة الأولى

لأننا نريد بالدرجة الأولى

لأننا نريد بالدرجة الأولى

$(x^2 + xy + y^2) dx - x^2 dy = 0$

$M_x = 2x \neq M_y = x + 2y$

$\div x^2 \rightarrow y' = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2 \rightarrow \text{Homog. } y' = F\left(\frac{y}{x}\right)$

Sol: $u = \frac{y}{x} \rightarrow y = xu \rightarrow y' = xu' + u$

$y' = xu' + u = F(u) = 1 + u + u^2$

$\hookrightarrow xu' + u = 1 + u + u^2$

$\hookrightarrow x \frac{du}{dx} = 1 + u^2 \rightarrow \frac{x}{dx} = \frac{1+u^2}{du} \Rightarrow \frac{1}{x} dx = \frac{1}{1+u^2} du$

$\Rightarrow \ln x + C = \tan^{-1} u$

$\hookrightarrow \tan(\ln x + C) = u \rightarrow y = x \tan(\ln x + C)$

Q) Find the value of m & n So that the D.E is Homog.

$(x^2 y^2 + \frac{1}{y^m}) \frac{dx}{dx} = x^n (\ln y^3 - \ln x^n) \frac{dy}{dy} \div dy + \text{homog}$

$x^2 y^2 + \frac{1}{y^m} = y' \rightarrow \frac{x^2 y^2 + \frac{1}{y^m}}{x^n (\ln y^3 - \ln x^n)} = 1 \rightarrow n=3$

$y' = \frac{x^2 \left(\frac{y}{x}\right)^2 + \frac{x^{-n}}{y^m}}{x^n \ln \left(\frac{y}{x}\right)^3}$

$\left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^4 \rightarrow m=-4$

وهو المطلوب

* The form $y' = G(ax+by)$

OR $y' = G(ax+by+c)$, $a \neq 0$, $b \neq 0$.

Ex) $y' = e^{2x+y}$ | $u = 2x+y$
 $G = e^u$

2 $y' = \sin(3x-y+4)$ | $u = 3x-y+4 \rightarrow \text{Form 2}$
 $G = \sin u$

Method 8: let $u = ax+by$ or $u = ax+by+c$

Transform D.E into Separable.

$$u' = a + by' \Rightarrow \frac{u' - a}{b} = y' \Rightarrow y' = \frac{1}{b} u' - \frac{a}{b}$$

In D.E $\frac{1}{b} u' + \frac{a}{b} = G(u) \rightarrow$ Solve by sep to find u

Use $u = ax+by$ or $u = ax+by+c$

(to find y)

Ex) Solve $y' = (x+y+3)^2 \rightarrow u = x+y+3$

$$x+y+3 = u$$

$$G = u^2$$

$$\rightarrow u' = 1 + y' \rightarrow \cancel{y' = u' - 1} \quad y' = u' - 1$$

$$\rightarrow u^2 = u' - 1 \rightarrow \frac{du}{dx} = u^2 - 1 \rightarrow \int \frac{1}{u^2 - 1} du = \int dx \quad \text{Sep in x \& u}$$

$$\rightarrow \tan^{-1} u = x + C \rightarrow u = \tan(x+C)$$

$$\rightarrow y + x + 3 = \tan(x+C)$$

$$\rightarrow y = \tan(x+C) - x - 3$$

Ex) Solve $y' = \sqrt{x^2 + 6xy + 9y^2} - \frac{1}{3}$, $y(0) = \frac{1}{3}$

* $y' = (x+3y)^{\frac{1}{2}} - \frac{1}{3}$

* $u = x + 3y$

* $G: u^{\frac{1}{2}} - \frac{1}{3}$

↳ $y' = G(ax+by)$

let $u = x + 3y \rightsquigarrow u' = 1 + 3y' \rightsquigarrow y' = \frac{1}{3}u' - \frac{1}{3}$

↳ $\frac{1}{3}u' - \frac{1}{3} = u^{\frac{1}{2}} - \frac{1}{3} \rightsquigarrow u' = 3u^{\frac{1}{2}}$

$\frac{du}{dx} = 3u^{\frac{1}{2}} \rightsquigarrow \int \frac{du}{3u^{\frac{1}{2}}} = \int dx$ $-\frac{2}{5} + \frac{5}{5} = \frac{3}{5}$

$x + C = \frac{1}{3} \cdot \frac{5}{3} \cdot u^{\frac{3}{2}}$

↳ $x + C = \frac{5}{9} u^{\frac{3}{2}}$

$$G(ax+by+c) = G(ax+by) \cdot G$$

Ex) Solve $y' + x+y+1 = (x+y)^2 e^{3x}$

$$* y' = (x+y)^2 e^{3x} - (x+y+1)$$

$$* \text{let } u = x+y \rightarrow u' = 1+y' \rightarrow y' = u' - 1$$

$$G = u^2 e^{3x} - u - 1$$

$$* u' - 1 = u^2 e^{3x} - u - 1$$

$$* u' = u^2 e^{3x} - u$$

$$* u' + u = u^2 e^{3x} \rightarrow \text{Bernoulli in } u$$

$$* v = u^{1-n} = u^{-1}$$

$$* \text{Linear in } v \rightarrow v' + (1+n)P(x)v = Q(x)$$

$$* v' + -v = -e^{3x} \quad f = -e^{3x}$$

$$* \text{I.F.} \rightarrow M(x) = e^{\int -1 dx} = e^{-x}$$

$$* v(x) = e^x \left[\int e^{-x} (-e^{3x}) dx + C \right]$$

$$* v(x) = e^x \left[\int -e^{2x} dx + C \right] \rightarrow = e^x \left[-\frac{e^{2x}}{2} + C \right]$$

$$* v(x) = -\frac{1}{2} e^{3x} + e^x C$$

$$* u^{-1} = -\frac{1}{2} e^{3x} + e^x C$$

$$* x+y = -\frac{1}{2} e^{3x} + e^x C \rightarrow y = -\frac{1}{2} e^{3x} + e^x C - x$$

Ex) Find the largest interval where IVP has unique sol.

$$y' + \frac{y}{\sin(2x) \ln(\sqrt[3]{x}-1)^2} = x, \quad y(-\frac{1}{2}) = 3$$

$$\text{I.V.P } y' = f(x,y), \quad y(x_0) = y_0$$

$$\frac{1}{f(x,y)} \text{ cont} \Leftrightarrow \frac{dy}{dx} \neq \text{cont}$$

$$* f(x,y) = x - \frac{y}{\sin(2x) \ln(\sqrt[3]{x}-1)^2} \rightarrow \text{discont. pts}$$

$$\textcircled{1} \sin(2x) = 0 \rightarrow 2x = 0, \pi, 2\pi \rightarrow x = 0, \frac{\pi}{2}, \pi, \dots$$

$$\textcircled{2} \ln(\sqrt[3]{x}-1)^2 = 0 \rightarrow (\sqrt[3]{x}-1)^2 = 1 \rightarrow \sqrt[3]{x}-1 = \pm 1 \rightarrow \sqrt[3]{x} = 2 \rightarrow x = 8$$

$$\textcircled{3} (\sqrt[3]{x}-1)^2 = 0 \rightarrow \sqrt[3]{x} = 1 \rightarrow x = 1$$

$$-\pi \quad -\frac{\pi}{2} \quad 0 \quad \frac{\pi}{2} \quad \pi$$

Interval of cont

$$\text{AS } x_0 = -\frac{1}{2} \in (-\frac{\pi}{2}, 0)$$

Ex) Show that $\cos(y-x) = Ce^{-x}$ where $C \in \mathbb{R}$ is
Sol. to D.E $\tan(y-x)(y'-1) - 1 = 0$

نقطة

$$-\sin(y-x)(y'-1) = -Ce^{-x}$$

$$\text{But } Ce^{-x} = \cos(y-x)$$

$$\text{إذن } \sin(y-x)(y'-1) = \cos(y-x)$$

$$\div \cos(y-x) \Rightarrow \tan(y-x)(y'-1) = 1$$

$$\tan(y-x)(y'-1) - 1 = 0 \quad \#$$

CH 4.8 Linear second O.D.E

1 The general form for second order

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = g(x)$$

Ex: $x^2 y'' + 2x y' + 2y = 0$

$$y'' + 4y' + 4y = 0$$

2 If $g(x) = 0 \Rightarrow$ D.E is Homog

If $g(x) \neq 0 \Rightarrow$ D.E is Not-Homog

3 I.V.P = D.E + I.C

an: $y'' + p(x)y' + q(x)y = f(x)$

with $y(x_0) = y_0, y'(x_0) = y_1 \Rightarrow$ constants $y_0, y_1 \in \mathbb{R}, x_0 = x_0 \in \mathbb{R}$

4 Exact and Unique

If the function $p(x), q(x)$ and $f(x)$ are cont on interval contains x_0 , then IVP has Unique Sol.

Ex) Find the largest interval where IVP has unique Sol

$$(1+x^2)y'' + x y' - y = \tan x$$

$$y'' + \frac{x}{1+x^2} y' - \frac{y}{1+x^2} = \frac{\tan x}{1+x^2}$$

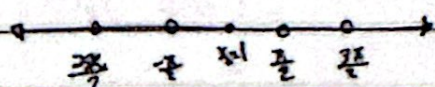
$y(1) = 0, y'(1) = \pi$

$p(x) \neq q(x)$ cont $\forall x \in \mathbb{R}$ but $f(x) = \frac{\sin x}{\cos x (1+x^2)}$

* $\cos x = 0 \Rightarrow x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \dots, \pm (2n+1) \frac{\pi}{2}$

$f(x,y)$ cont on $\mathbb{R} \setminus \{x = \pm (2n+1) \frac{\pi}{2}\}$

Interval of cont



As $x_0 = 1 \in (-\frac{\pi}{2}, \frac{\pi}{2}) \Rightarrow$ largest interval $(-\frac{\pi}{2}, \frac{\pi}{2})$

in IC

$$f(x) = x+1$$

$$g(x) = 2x+2 = 2f(x)$$

$$\frac{g}{f} = 2 \rightarrow \text{const}$$

Defn: Two function $f(x)$ and $g(x)$ are linearly dependent on

$$\text{if } \frac{f(x)}{g(x)} = C$$

If $\frac{f(x)}{g(x)} \neq \text{constant} \Rightarrow f(x)$ and $g(x)$ are linear indep

Ex) ① $f(x) = e^{2x}$, $g(x) = e^{5x}$

$$\frac{f}{g} = \frac{e^{2x}}{e^{5x}} = e^{-3x} \neq C \quad \text{linear indep}$$

② $f(x) = \sin 2x$, $g(x) = \sin x \cos x$

$$\frac{f}{g} = \frac{\sin 2x}{\frac{1}{2} \sin 2x} = 2 \rightarrow \text{linear dep}$$

* Matrix $n \times m$

$$A_{\substack{n \times m \\ \text{row}}} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \begin{matrix} \text{row} \\ \text{m: column} \end{matrix}$$

$$* \det(A) = |A| = a_{11}a_{22} - a_{12}a_{21}$$

Defn: Wronskian of two function f, g

$$W \begin{bmatrix} f(x) & g(x) \end{bmatrix} = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = f(x)g'(x) - g(x)f'(x)$$

Dep $\Rightarrow$ Wronskian = 0

$$W[s, g, h] = \begin{matrix} s & g & h \\ s' & g' & h' \\ s'' & g'' & h'' \end{matrix}$$

$$\begin{matrix} s & | & g' & h' & | & -g & | & s' & h' & | & +h & | & s' & g' & | \\ & | & g'' & h'' & | & & | & s'' & h'' & | & & | & s'' & g'' & | \end{matrix}$$

Second ODE

$$y'' + p(x)y' + q(x)y = f(x), \quad y(x_0) = y_0, \quad y'(x_0) = y_1$$

(y_0, y_1) constant, x_0 fixed

* order of D.E. $\leadsto$ # of I.C.

* If $p(x)$, $q(x)$ and $f(x)$ are cont on I , contains x_0
 $\hookrightarrow$ Then, I.V.P has Unique Sol.

Note: If y_1 and y_2 are two Sol to D.E.

$$ay'' + by' + cy = 0 \quad \text{--- (1)}$$

Then = general Sol. is

$$y(x) = C_1 y_1 + C_2 y_2$$

any linear combination of Sol is also Sol. Second order

Q) If y_1 and y_2 is Sol D.E (1) $\Rightarrow y = C_1 y_1 + C_2 y_2$ is also Sol.

$$* y' = C_1 y_1' + C_2 y_2' \quad / \quad * y'' = C_1 y_1'' + C_2 y_2''$$

$$\text{in D.E } a(C_1 y_1' + C_2 y_2') + b(C_1 y_1 + C_2 y_2) + C(C_1 y_1 + C_2 y_2) \stackrel{?}{=} 0$$

$$C_1 (ay_1' + by_1 + cy_1) + C_2 (ay_2' + by_2 + cy_2) \stackrel{?}{=} 0$$

= zero as y_1 is Sol

= zero as y_2 is Sol

$$C_1 \neq 0 + C_2 \neq 0 = 0 \quad \#$$

$$W[f(x), g(x)] = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = f g' - g f'$$

Note: ① If $W[f(x), g(x)] = 0 \quad \forall x \in I$

Then f, g are linear-dep.

② If $W[f(x), g(x)] \neq 0 \quad \forall x \in I$

Then f, g are linear-indep.

* Find Wronskian &

$$f(x) = \sin 2x$$

$$g(x) = \sin x \cos x \rightarrow = \frac{1}{2} \sin 2x$$

$$W[f, g] = \begin{vmatrix} \sin 2x & \cancel{\frac{1}{2} \sin 2x} \\ 2\cos 2x & \cos 2x \end{vmatrix} = \sin 2x \cos 2x - \sin 2x \cos 2x = 0$$

$W=0 \rightarrow f, g$ lin. dep

Defn: Fundamental Set of Sol (F.S.S)

For D.E $ay'' + by' + cy = 0$, we say that the set $\{y_1, y_2\}$ is F.S.S. I.S.S

① y_1 and y_2 are sol for D.E

② y_1 and y_2 are lin. indep $\rightarrow (W[y_1, y_2] \neq 0)$

Note: If $\{y_1, y_2\}$ is F.S.S Then, general Sol is
 $y(x) = C_1 y_1 + C_2 y_2$

Ex) Show that $\{e^{2x}, e^{-2x}\}$ forms I.S.S. for D.E
 $y'' - 4y = 0$

$$* y_1 = e^{2x} \rightarrow y' = 2e^{2x} \rightarrow y'' = 4e^{2x}$$

$$* y_2 = e^{-2x} \rightarrow y' = -2e^{-2x} \rightarrow y'' = +4e^{-2x}$$

$$4e^{2x} - 4e^{2x} = 0$$

$$+4e^{-2x} - 4e^{-2x} = 0$$

$\rightarrow y_1$ and y_2 is Sol

$$\& \text{lin. indep} \rightarrow W[e^{2x}, e^{-2x}] = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -2 - 2 = -4 \neq 0$$

$W \neq 0 \rightarrow y_1 = e^{2x}$ and $y_2 = e^{-2x}$ is lin. indep.

Now $\rightarrow \{e^{2x}, e^{-2x}\}$ is F.S.S

Reduction of order 8.

$$\begin{cases} y'' + P(x)y' + Q(x)y = 0 \\ \text{and } y_1(x) \neq 0 \text{ is Sol. (given) for D.E} \end{cases}$$

* ~~Let~~ $y_1(x)$ is Sol. $y_1(x) = ?$

$$* \frac{y_2(x)}{y_1(x)} = v(x)$$

* As y_1 and y_2 is lin. indep. let $y_2(x) = v(x)y_1(x)$

* y_2 is Sol. Satisfy D.E

$$* y_2 = v \cdot y_1 \rightarrow y_2' = v \cdot y_1' + y_1 v'$$

$$* y_2'' = v \cdot y_1'' + y_1' v' + y_1 v'' + v' y_1' \rightarrow y_2'' = v \cdot y_1'' + 2v' y_1' + v'' y_1$$

in D.E 8

$$\rightarrow v y_1'' + 2v' y_1' + v'' y_1 + P(x)(v y_1' + y_1 v') + Q(x)(v y_1) = 0$$

$v(y_1'' + P(x)y_1' + Q(x)y_1)$ zero because y_1 is Sol.

$$\therefore 2v' y_1' + v'' y_1 + P(x) y_1 v' = 0$$

$$(y_1 \neq 0) \rightarrow 2v' y_1' = -y_1(v'' + P(x)v')$$

$$\rightarrow \int \frac{2v'}{y_1} = \int \left(\frac{v''}{y_1} + P(x) \right) dx$$

Solve for v

$$2 \ln y_1 = -(\ln v' + \int P(x) dx)$$

$$y_1^2 = v'^{-1} \cdot e^{-\int P(x) dx}$$

$$\rightarrow v(x) = \int \frac{1}{y_1^2} e^{-\int P(x) dx} dx$$

$$F.S.S = \{y_1, y_2\}$$

↳ lin. indep

$$\frac{y_2}{y_1} \neq \text{constant} \quad \text{OR} \quad W[y_1, y_2] \neq 0$$

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1' \neq 0$$

∴ y_1 and y_2 is Sol.

$$* y'' + P(x)y' + Q(x)y = S(x)$$

If $S(x) = 0 \rightarrow$ Homog

If $S(x)$ is function of $x \rightarrow$ non Homog

* Reduction of order.

one Sol. is given $[y_1]$

$$y'' + \underbrace{P(x)}_{x\text{-only}} y' + \underbrace{Q(x)}_{x\text{-only}} y = 0$$

(1) let

$$* \text{ let } y_2 = V(x) y_1$$

$$V(x) = \int \frac{1}{y_1^2} e^{-\int P(x) dx} dx$$

$$y_2 = y_1 \left[\int \frac{1}{y_1^2} e^{-\int P(x) dx} dx \right]$$

Where $P(x)$ is coeff. y' in D.E

Ex) If $y_1 = e^{3x}$ is Sol to $y'' - \left(\frac{3}{x} + 6\right)y' + 9(x)y = 0, x > 0$
Find the general Sol.

$$y_2 = e^{3x} \left[\frac{1}{e^{6x}} e^{-\int (\frac{3}{x} + 6) dx} dx \right] = e^{3x} \left[\frac{1}{e^{6x}} e^{-3\ln x - 6x} \right] = e^{3x} \cdot e^{-6x} \cdot e^{-3\ln x}$$

$$y_2 = e^{3x} \int x^{-3} dx \Rightarrow e^{3x} \frac{x^{-2}}{-2}$$

$$\text{General Sol} : y(x) = C_1 e^{3x} + C_2 e^{3x} x^{-2}$$

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

* Abel's Formula :

→ If y_1 and y_2 are Sol. to D.E

$$y'' + p(x)y' + q(x)y = 0, \text{ then } \&$$

$$W[y_1, y_2]_{x_0} = C e^{-\int p(x) dx}$$

$C \in \mathbb{R}$ arbitrary constant

الشيء

$$W' = C e^{-\int p(x) dx} \cdot (-p(x))$$

$$W' = W(-p(x))$$

$$p(x) = -\frac{W'}{W}$$

نقول ان الشيء ونشتق

فستكون $e^{-\int p(x) dx}$ مضروب في C ويكون ذلك هو الحل

هذا الشيء يبقى فاد ان كان $p(x)$ ثابت

Ex) Let y_1, y_2 are Sol. for

$$x^2 y'' - 2y' + (3+x)y = 0, \quad x > 0$$

Find $W[y_1, y_2](x)$

$$\text{Def } y'' - \frac{2}{x^2} y' + \frac{(3+x)}{x^2} y = 0$$

$$W = C e^{-\int p(x) dx} = C e^{-\int -\frac{2}{x^2} dx} = C e^{-2x^{-1}}$$

2 If $W[y_1, y_2](2) = 3$, Find $W[y_1, y_2](5)$.

→ To Find C Use $W(2) = 3$

$$C e^{-2 \cdot 2^{-1}} = 3 \Rightarrow C = 3e$$

$$W[y_1, y_2](5) = 3e * e^{-2 \cdot 5^{-1}} = 3e^{\frac{2}{5}}$$

Ex) Let y_1, y_2 two sol to the D.E

$y'' + P(x)y' + Q(x)y = 0$, with x is one sol and

$$W[y_1, y_2](x) = x^3 e^x.$$

Find the general sol.

$$P(x) = -\left(\frac{x^3 e^x + e^x + 3x^2}{x^3 e^x}\right) = -\left(1 + \frac{3}{x}\right)$$

$$y_2 = y_1 \left[\int \frac{1}{y_1^2} e^{-\int P(x) dx} dx \right]$$

$$= x \left[\int \frac{1}{x^2} e^{\int (1 + \frac{3}{x}) dx} dx \right] \leadsto x \left[\int \frac{1}{x^2} e^x \cdot x^3 dx \right]$$

$$y_2 = x \int x e^x dx$$

$$\leadsto = x(xe^x - e^x) \leadsto x^2 e^x - x e^x$$

$$* \text{ General Sol : } y(x) = C_1 x + C_2 (x^2 e^x - x e^x)$$

7-11-2023

(14) 6/13

Second order D.E (Homog)

I Reduction of order

→ y_1 given

$$y'' + P(x)y' + Q(x)y = 0$$

$$* y_2 = v(x) y_1$$

II Second order Linear D.E with constant coefficient

$$a y'' + b y' + c y = 0 \text{ D.E}$$

$a \neq 0$ / a, b, c are constant.

$$\geq 3y'' + y = 0 \rightarrow (\text{S.O.L.D.E.W.C.C})$$

$$a=3, b=0, c=1$$

$$\geq x^2 y'' + y' = 0 \quad x \rightarrow \text{because } a=x^2 \text{ (not const)}$$

$$* \text{ Sol } y = e^{rx}$$

→ with constant coeff

→ where r is the root of characteristic eqn.

As $(y = e^{rx})$ is Sol $\Rightarrow$ Satisfy D.E

$$y' = r e^{rx}, \quad y'' = r^2 e^{rx}$$

in D.E

$$a r^2 e^{rx} + b r e^{rx} + c e^{rx} = 0$$

$$e^{rx} (a r^2 + b r + c) = 0$$

$$\Rightarrow \text{char. eqn : } a r^2 + b r + c = 0$$

$$\Rightarrow y'' + p y' + q y = 0, \quad y' = r, \quad y = 1$$

$y^{(n)} \rightarrow r^n$
derivative $\rightarrow$ Power

$$ar^2 + br + c = 0$$

Find root for this eqn.

$$\Delta = b^2 - 4ac$$

$$r = \frac{-b \pm \sqrt{\Delta}}{2a}$$

Case 1) If $\Delta > 0$

There are Two different root

$$r_1 \neq r_2$$

$$F.S.S = \{ y_1 = e^{r_1 x}, y_2 = e^{r_2 x} \}$$

General Sol:

$$y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

Case 2) If $\Delta = 0$

There are Repeated Root

$$r_1 = r_2 = r$$

$$r = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-b}{2a}$$

$$F.S.S = \{ y_1 = e^{rx}, y_2 = x e^{rx} \} \rightarrow \frac{y_1}{y_2} = \frac{e^{rx}}{x e^{rx}} = \frac{1}{x} \rightarrow \text{con}$$

So y_1, y_2 are lin dep x

نحتاج إلى إيجاد الحل الثاني
الذي لا

* Use Reduction of order to find y_2

$$y_1 = e^{rx} \rightarrow r = \frac{-b}{2a} \rightarrow e^{\frac{-b}{2a}x} \rightarrow e^{\frac{-b}{2a}x}$$

$$ay'' + by' + cy = 0 \rightarrow y'' + \frac{b}{a}y' + \frac{c}{a}y = 0 \quad (y_2 = v_2 y_1 \text{ jeo})$$

$$v_2 = \int \frac{1}{y_1^2} e^{\int \frac{b}{a} dx} dx = \int \frac{1}{e^{\frac{b}{a}x}} e^{\int \frac{b}{a} dx} dx$$

$$= \int \frac{1}{e^{\frac{b}{a}x}} e^{\frac{b}{a}x} dx = \boxed{x}$$

$$y_2 = x e^{rx}$$

So: $y_1 = e^{rx}$ $y_2 = e^{rx} * x$

General Sol:

$$y_{gen} = C_1 e^{rx} + C_2 x e^{rx}$$

Case 3) complex number:

$$i = \sqrt{-1} \rightarrow i^2 = -1$$

$$* \sqrt{-4} = \sqrt{-1} \cdot \sqrt{4} = \sqrt{-1} \cdot 2 = 2i$$

$$* \sqrt{-16} = 4i$$

Complex number:

$$Z = \alpha \mp i\beta$$

α : is real part

β : is imaginary part

Case 3) If $\Delta < 0$

Then root is complex.

$$r = \frac{-b \pm \sqrt{\Delta}}{2a}$$

$$ay'' + by' + cy = 0$$

$$\rightarrow ar^2 + br + c = 0$$

$$r_{1,2} = \alpha \mp i\beta$$

$$y = e^{rx} = e^{(\alpha + i\beta)x} = e^{\alpha x + i\beta x} = e^{\alpha x} * e^{i\beta x}$$

هنا عايز نعرف على i ولا يجوز في F.S.S

$$\mp e^{i\theta} = \cos \theta \mp i \sin \theta$$

يكون في i بالتالي: - نستخرج من المعادلة

$$y_1 = e^{\alpha x} * e^{i\beta x} = e^{\alpha x} [\cos \beta x + i \sin \beta x]$$

$$F.S.S = \{ \underbrace{e^{\alpha x} \cos \beta x}_{y_1} \bullet \underbrace{e^{\alpha x} \sin \beta x}_{y_2} \}$$

الـ C_1 و C_2 هما ثابتان
(General Soln) الموجود

General Sol:

$$y = C_1 y_1 + C_2 y_2$$

ر.ج. الـ r

$$r = -2 \pm 4i = -1 \pm 2i$$

$$\alpha = -1 / \beta = 2$$

9-11-2023

(15) Ques 12

$$ay'' + by' + cy = 0$$

$$y^{(n)} \rightarrow r^n$$

$$y = e^{rx}$$

$$\text{char. eqn: } ar^2 + br + c = 0$$

Find root for char. eqn (C)

$$\Delta = b^2 - 4ac$$

cases	Root for char	F.S.S.
$\Delta > 0$	$r_1 \neq r_2$	$e^{r_1 x}, e^{r_2 x}$
$\Delta = 0$	$r_1 = r_2 = r$	$e^{rx}, x e^{rx}$
$\Delta < 0$	$r = \alpha \pm i\beta$ ↳ complex root	$e^{\alpha x} \cos(\beta x), e^{\alpha x} \sin(\beta x)$

* Ex: Solve the following D.E

$$① y'' - 5y' + 6y = 0 \rightarrow \text{Homog}$$

$$a=1, b=-5, c=6$$

$$\text{char. eqn: } r^2 - 5r + 6 = 0$$

$$\Delta = 25 - 4 \times 1 \times 6 = 1 > 0$$

$$(r-2)(r-3) = 0$$

$$r=2 \quad \text{or} \quad r=3$$

$$\text{F.S.S} = \{e^{2x}, e^{3x}\}$$

$$\text{general} \rightarrow y = C_1 e^{2x} + C_2 e^{3x}$$

$$② y'' - 4y' + 4y = 0$$

$$a=1, b=-4, c=4$$

$$\text{char. eqn: } r^2 - 4r + 4 = 0$$

$$\Delta = 16 - 4 \times 1 \times 4 = 0$$

$$(r-2)^2 = 0 \rightarrow r=2$$

$$r_1 = r_2 = r = 2$$

general Soln:

$$y(x) = C_1 e^{2x} + C_2 x e^{2x}$$

Q] If $y = C_1 e^{2x} + C_2 x e^{2x}$

is Sol for D.E, write D.E.

$y_1 = e^{2x}$, $y_2 = x e^{2x}$

$r = r_1 = r_2 = 2$

$(r-2)(r-2) = 0$
 $\hookrightarrow \text{char. eqn} : (r-2)^2 = 0$

$\hookrightarrow r^2 - 4r + 4 = 0$

$\hookrightarrow y'' - 4y' + 4y = 0$ ✗

$y^{(n)} \longleftrightarrow r^n$

[3] $y'' + 16y = 0$

$a=1$, $b=0$, $C=16$

char. eqn : $r^2 + 16 = 0$

$\Delta = 0 - 4 \times 1 \times 16 = -64 < 0$

$r = \frac{-0 \pm \sqrt{-64}}{2 \times 1} = \frac{0 \pm 8i}{2} = 0 \pm 4i$
 $\hookrightarrow \alpha = 0, \beta = 4$

F.S.S = $\{ e^{0x} \cos(4x), e^{0x} \sin(4x) \}$

General Sol : $y(x) = C_1 \cos(4x) + C_2 \sin(4x)$

[Ex] Solve : $y'' - 2y' + 4y = 0$, $y(0) = 0$, $y'(0) = 2$

$a=1$, $b=-2$, $C=4$

char. eqn : $r^2 - 2r + 4 = 0$

$\Delta = 4 - 4 \times 1 \times 4 = -12 < 0$

Complex Root : $r = \frac{+2 \pm \sqrt{-12}}{2 \times 1} = +1 \pm i\sqrt{3}$
 $\hookrightarrow \alpha = 1, \beta = \sqrt{3}$

General Sol : $y(x) = C_1 e^x \cos(\sqrt{3}x) + C_2 e^x \sin(\sqrt{3}x)$

$\hookrightarrow y(0) = 0 \Rightarrow 0 = C_1 + 1 \times 1 + 0$

$C_1 = -1$

$y = C_2 e^x \sin(\sqrt{3}x) \Rightarrow y' = C_2 (e^x \times \sqrt{3} \cos(\sqrt{3}x) + \sin(\sqrt{3}x) \times e^x)$

$y'(0) = 2 \Rightarrow 2 = C_2 (1 \times \sqrt{3} \times 1 + 0) \Rightarrow C_2 = \frac{2}{\sqrt{3}}$

General Sol : $y(x) = \frac{2}{\sqrt{3}} e^x \sin(\sqrt{3}x)$

Q) If the roots of the Auxiliary ^{char} equation are

$$r_{1,2} = 3 \pm i \quad \text{Find D.E}$$

$$\alpha = 3, \beta = 1$$

$$y(x) = C_1 e^{3x} \cos(x) + C_2 e^{3x} \sin(x)$$

$$\text{let } r = 3 \pm i$$

$$i = \sqrt{-1}$$

$$\text{دعونا نترضف } \rightarrow r - 3 = \pm i$$

$$i^2 = -1$$

$$(r-3)^2 = -1$$

$$\rightarrow r^2 - 6r + 9 = -1 \rightarrow r^2 - 6r + 10 = 0$$

$$r^n \rightarrow y^{(n)}$$

$$\rightarrow \text{D.E} \rightarrow y'' - 6y' + 10y = 0$$

Q) If $y_1 = e^x$ and $y_2 = e^{-3x}$

are Soln to D.E :

$$ay'' + (b-1)y' - 5cy = 0, \text{ Find } a, b, c.$$

$$r_1 = 2, \quad r_2 = -3$$

$$\rightarrow r-2=0, \rightarrow r+3=0$$

$$\text{Char. eqn : } (r-2)(r+3) = 0$$

$$\rightarrow r^2 + r - 6 = 0$$

$$\rightarrow \text{D.E} \rightarrow y'' + y' - 6y = 0$$

$$* \boxed{a=1}$$

$$* b-1 = 1 \rightarrow \boxed{b=2}$$

$$* -5c = -6 \rightarrow \boxed{c = \frac{6}{5}}$$

14-11-2023

Al-Jalal

⊥ Reduction of order a

$$y'' + P(x)y' + Q(x)y = 0$$

Given y_1

$$y_2 = V(x)y_1 = y_1 \int \frac{1}{y_1^2} e^{-\int P(x)dx} dx$$

≡ constant coeff

$$ay'' + by' + cy = 0$$

$$\text{Sol: } y = e^{rx} \quad (y^{(n)} \rightarrow r^n)$$

$$ar^2 + br + c = 0$$

≡ Cauchy Euler D.E

↳ All terms in D.E has same $(x^n y^{(n)})$

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \dots + a_1 x y' + a_0 y = 0$$

* Second order Cauchy Euler:

$$a \neq 0 \quad ax^2 y'' + bxy' + cy = 0 \quad \Rightarrow a, b, c \text{ are constant}$$

Ex

$$\perp 3x^2 y'' + 2xy' + y = 0 \quad \Rightarrow a=3, b=2, c=1 \quad (\text{Cauchy Euler's Second order Homog})$$

$$\equiv x^2 y'' + xy = 0 \quad \Rightarrow \text{Not Cauchy}$$

is Sol $(y = x^r)$ Satisfy D.E

$$y' = r x^{r-1}$$

$$y'' = r(r-1) x^{r-2}$$

$$\text{in D.E } \Rightarrow ax^2 y'' + bxy' + cy = 0$$

$$\rightarrow ax^2 r(r-1) x^{r-2} + b x r x^{r-1} + c x^r = 0$$

$$\rightarrow ar(r-1) x^r + br x^r + c x^r = 0$$

$$\rightarrow x^r (ar^2 - ar + br + c) = 0$$

$$\text{char eqn: } ar^2 - ar + br + c = 0$$

$$\rightarrow ar^2 + (b-a)r + c = 0$$

Char. eqn: $ar^2 + (b-a)r + c = 0$

$\Delta = (b-a)^2 - 4ac$

Case 1: $\Delta > 0$

$\hookrightarrow r_1 \neq r_2$ two different roots

F.S.S = $\{x^{r_1}, x^{r_2}\}$

$\hookrightarrow$ general Sol $y(x) = C_1 x^{r_1} + C_2 x^{r_2}$

Case 2: $\Delta = 0$

$\hookrightarrow$ Repeated root $r_1 = r_2 = r$

* $y_1 = x^r$ $y_2 = ?$

$\hookrightarrow ax'' + by' + cy = 0, x > 0$

$\div a \Rightarrow y'' + \frac{b}{a}y' + \frac{c}{a}y = 0$

$\hookrightarrow$ root for char. eqn.

$\Delta = 0, r = \frac{-(b-a) \pm \sqrt{\Delta}}{2a} \rightarrow r = \frac{-(b-a)}{2a} \rightarrow 2r = -\frac{b}{a} + 1$

$y_2 = \int \frac{1}{x^r} e^{-\int \frac{b}{a} \frac{1}{x} dx} dx$

$e^{-\frac{b}{a} \ln x} = e^{\ln x^{-\frac{b}{a}}} = x^{-\frac{b}{a}}$

$= \int \frac{1}{x^{\frac{b}{a}+1}} e^{\frac{b}{a} \ln x} dx$

$= \int \frac{1}{x^{\frac{b}{a}+1}} x^{\frac{b}{a}} dx = \int \frac{1}{x} dx = \ln x$

$y_2 = x^r \ln x$

Case 3: $\Delta < 0$ complex root

$r = \frac{-(b-a) \pm \sqrt{\Delta}}{2a} \Rightarrow r_{1,2} = \alpha \pm i\beta$

$y = x^r = x^{\alpha + i\beta} \Rightarrow y = x^\alpha x^{i\beta} = x^\alpha \cdot e^{i\beta \ln x}$

$\Rightarrow y = x^\alpha [e^{i\beta \ln x}] = x^\alpha [\cos(\beta \ln x) + i \sin(\beta \ln x)]$

F.S.S = $\{x^\alpha \cos(\beta \ln x), x^\alpha \sin(\beta \ln x)\}$

Ex) Solve

$$\underline{\underline{1}} \quad 3x^2 y'' + 11xy' - 3y = 0, \quad x > 0$$

$a=3, b=11, c=-3$ $\rightarrow$ Homog. second order, Cauchy-Euler

char. eqn:

$$ar^2 + (b-a)r + c = 0$$

$$3r^2 + 8r - 3 = 0 \quad \rightarrow \quad \Delta = 64 - 4 \cdot 3 \cdot (-3) = 100 > 0$$

$$r = \frac{-8 \pm 10}{6} \quad \rightarrow \quad r_1 = \frac{-8+10}{6} = \frac{2}{6} = \frac{1}{3}$$

$$r_2 = \frac{-8-10}{6} = \frac{-18}{6} = -3$$

$$F.S.S. = \{x^{\frac{1}{3}}, x^{-3}\}$$

$$\text{General sol} \rightarrow y(x) = C_1 x^{\frac{1}{3}} + C_2 x^{-3}$$

$$\underline{\underline{2}} \quad x^2 y'' + xy' = 0$$

$$a=1, b=1, c=0$$

char. eqn:

$$r^2 + 0r + 0 = 0 \Rightarrow r^2 = 0$$

$$r_1 = r_2 = 0$$

$$F.S.S. = \{1, \ln x\}$$

$$y(x) = C_1 + C_2 \ln x$$

16-11-2023

(17) Syllabus

Cauchy Euler Equation: $x^n y^{(n)}$

$$ax^2 y'' + bxy' + cy = 0$$

 $a \neq 0$, a, b, c are constant.

$$y = x^r$$

$$\text{Char eqn: } ar^2 + (b-a)r + c = 0$$

$$\Delta = (b-a)^2 - 4ac$$

Case	Root of char	F.S.S
$\Delta > 0$	$r_1 \neq r_2$	x^{r_1}, x^{r_2}
$\Delta = 0$	$r_1 = r_2 = r$	$x^r, x^r \ln x$
$\Delta < 0$	complex root $r = \frac{(b-a) \pm i\sqrt{\Delta}}{2a}$ $= \alpha \pm i\beta$	$x^\alpha \cos(\beta \ln x),$ $x^\alpha \sin(\beta \ln x)$

Ex) Solve: $x^2 y'' + 5xy' + 5y = 0$, $x > 0$ Cauchy Euler
 $a=1, b=5, c=5$

$$\text{char eqn} \rightarrow r^2 + 4r + 5 = 0$$

$$\Delta = 16 - 4 \times 5 \times 1 = 16 - 20 = -4$$

$$r = \frac{-4 \pm \sqrt{-4}}{2} = -2 \pm i \Rightarrow \alpha = -2, \beta = 1$$

$$y(x) = C_1 x^{-2} \cos(\ln x) + C_2 x^{-2} \sin(\ln x)$$

Ex) Solve: $x^2 y'' - 4xy' + 4y = 0$ with $(y(1) = -2, y'(1) = -11)$

$$a=1, b=-4, c=4, (b-a) = -5$$

$$\Delta = 25 - 4 \times 4 \times 1 = 9 > 0$$

$$\text{char. eqn} \rightarrow r^2 - 5r + 4 = 0$$

$$(r-4)(r-1) = 0$$

$$r=4, r=1$$

$$y(x) = C_1 x^4 + C_2 x^1, \text{ Use I.C. to find } C_1, C_2$$

$$-2 = C_1 + C_2 \dots (1)$$

$$y'(x) = 4x^3 C_1 + C_2$$

$$-11 = 4C_1 + C_2 \dots (2)$$

$$(1)-(2) \Rightarrow 9 = -3C_1$$

$$\left. \begin{array}{l} C_1 = -3 \Rightarrow C_2 = 1 \\ y(x) = -3x^4 + x \end{array} \right\}$$

$$y(x) = -3x^4 + x$$

Q.) Let $y = x^3 \ln x$ be a sol of DE

$$Ax^2 y'' + Bxy' + Cy = 0$$

① Show that $C = 9A$ P

② Find the values of A, B, C P

$\hookrightarrow$ AS $y = x^3 \ln x$ is sol $\Rightarrow$ Satisfy DE

$$y' = x^2 \frac{1}{x} + \ln x \cdot 3x^2 = x^2 + 3x^2 \ln x = x^2(1 + 3 \ln x)$$

$$y'' = x^2(0 + 3 \frac{1}{x}) + (1 + 3 \ln x) \cdot 2x = 3x + 2x + 6x \ln x = 5x + 6x \ln x$$

In D.E

$$Ax^2(5x + 6x \ln x) + bX(x^2 + 3x^2 \ln x) + Cx^3 \ln x = 0$$

$$5Ax^3 + 6Ax^3 \ln x + bx^3 + 3bx^3 \ln x + Cx^3 \ln x = 0$$

$$x^3(5A + b) + x^3 \ln x(6A + 3b + C) = 0$$

$$5A + b = 0 \Rightarrow b = -5A$$

$$6A + 3b + C = 0 \Rightarrow 6A + 3(-5A) + C = 0$$

$$6A - 15A + C = 0 \Rightarrow -9A + C = 0 \Rightarrow 9A = C$$

$\hookrightarrow$ AS $y = x^3 \ln x \Rightarrow r = 3$ is repeated root

$$r_1 = r_2 = 3$$

$$\text{char. eqn: } (r-3)^2 = 0$$

$$r^2 - 6r + 9 = 0$$

$$ar^2 + (b-a)r + c = 0 \Rightarrow a=1, b-a=-6 \Rightarrow b=-5$$

$$\Rightarrow x^2 y'' - 5xy' + 9y = 0$$

$$C=9$$

EX) If $y = C_1 x^a + C_2 x^b \ln x$, is general sol to D.E

$$x^2 y'' - 6xy' + 8ky = 0, \text{ where } \{k, b\} \text{ are const}$$

Find $\{k, b\}$.

$$r_1 = r_2 = 2 \Rightarrow r - 2 = 0 \Rightarrow r^2 - 4r + 4 = 0$$

$$xy'' - 3xy' + 4y = 0$$

$$-6b = -3 \quad \& \quad 8k = 4$$

$$b = \frac{1}{2}$$

$$k = \frac{1}{2}$$

$$a=1$$

$$C_1 = +4$$

$$b-a = -4$$

$$b = -3$$

Q) Solve: $x(x \cdot y')' + 25y = 0$

$x(x \cdot y'' + y' \cdot 1) + 25y = 0$

$x^2 y'' + x y' + 25y = 0 \Rightarrow$ Cauchy Euler

$a=1, b=1, c=25, (b-a)=0$

Char. eqn: $r^2 - ar + 25 = 0 \Rightarrow r^2 + 25 = 0$

$r = \pm 5i \Rightarrow r = 0 \pm 5i \Rightarrow \alpha=0, \beta=5$

$y(x) = C_1 \cos(5 \ln x) + C_2 \sin(5 \ln x)$

Q2) $(t+1)^2 y'' + 10(t+1)y' + 14y = 0$

~~$(t+1)^2 y'' + 10(t+1)y' + 14y = 0$~~

$x = t+1$

$x^2 y'' + 10xy' + 14y = 0$

$a=1, b=10, c=14$

$\Delta = 81 - 4 \cdot 14 + 1 = 25$

$r = \frac{9 \pm 5}{2} \Rightarrow r_1 = \frac{14}{2} = 7$

$r_2 = \frac{4}{2} = 2$

S.S.S = $\{x^7, x^2\}$

General Sol $\rightarrow y(x) = C_1 x^7 + C_2 x^2$

$y(t) = C_1 (t+1)^7 + C_2 (t+1)^2$ (t) is

Homog. Second ODE

[1] Reduction of order

 $\rightarrow y_1$ is given

$$\rightarrow y'' + B(x)y' + Q(x)y = 0$$

$$\rightarrow y_2 = v(x) y_1$$

$$\rightarrow v(x) = \int \frac{1}{y_1^2} e^{-\int B(x) dx} dx$$

[2] Constant coeff

$$\rightarrow ay'' + by' + cy = 0$$

char. eqn:

$$\rightarrow ar^2 + br + c = 0$$

Soh: $y = e^{rx}$

[1] $\Delta > 0 \rightarrow r_1 \neq r_2 \rightarrow \text{F.S.S} = \{e^{r_1 x}, e^{r_2 x}\}$

[2] $\Delta = 0 \rightarrow r_1 = r_2 = r \rightarrow \text{F.S.S} = \{e^{rx}, x e^{rx}\}$

[3] $\Delta < 0 \rightarrow \text{complex root}$

$$\rightarrow r = \frac{-b \pm \sqrt{\Delta}}{2a} = \alpha \pm i\beta$$

$$\text{F.S.S} = \{e^{\alpha x} \cos(\beta x), e^{\alpha x} \sin(\beta x)\}$$

[3] Cauchy Euler:

$$\rightarrow ax^2 y'' + bxy' + cy = 0$$

char. eqn:

$$ar^2 + (b-a)r + c = 0$$

$$b, a \rightarrow [x^n y^{(n)}]$$

Soh: $y = x^r$

[1] $\Delta > 0 \rightarrow r_1 \neq r_2 \rightarrow \text{F.S.S} = \{x^{r_1}, x^{r_2}\}$

[2] $\Delta = 0 \rightarrow r_1 = r_2 = r \rightarrow \text{F.S.S} = \{x^r, \ln x \cdot x^r\}$

[3] $\Delta < 0 \rightarrow \text{complex root}$

$$\rightarrow r = \alpha \pm i\beta$$

$$\text{F.S.S} = \{x^{\alpha} \cos(\beta \ln x), x^{\alpha} \sin(\beta \ln x)\}$$

Non-Homog. ODE

$$\hookrightarrow ay'' + by' + cy = f(x)$$

where $f(x) \neq 0$

general Soln for any Non-Homog D.E

$$\hookrightarrow y(x) = y_h + y_p$$

where y_h is homog Soln

$$y_h = C_1 y_1 + C_2 y_2$$

* y_p is called Particular Soln (Non-Homog-Soln).

* The Method of undetermined coefficient

$$ay'' + by' + cy = f(x)$$

where $f(x)$ has one of the following forms

- ① e^{ax+b} , (a,b) any constant
- ② Polynomial of degree n : $P_n(x)$
- ③ $\cos(ax+b)$ or $\sin(ax+b)$

or $\neq$ between them.

Note to find the form of $y_p(x)$

$f(x)$	$y_p(x)$
① e^{ax+b}	Ae^{ax+b}
② $P_n(x)$	all terms of Poly of degree n $P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$
③ $\cos \theta$ or $\sin \theta$	$A \sin \theta + B \cos \theta$

Ex Find the form of $y_p(x)$ for the following function

① $f(x) = e^{-3x} \rightarrow y_p(x) = Ae^{-3x}$

degree 4 $\text{② } f(x) = x^4 + 2x \rightarrow y_p(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0$

③ $f(x) = \sin(2x) \rightarrow y_p(x) = A \sin(2x) + B \cos(2x)$

$f(x) = \sin(3x) + \cos(5x) \rightarrow$

$y_p(x) = A \sin(3x) + B \cos(3x) + C \sin(5x) + D \cos(5x)$

Method 8

$$ay'' + by' + cy = f(x)$$

[1] Find the Homog Sol. $f(x) = 0 \rightarrow$ F.S.S. = $\{y_1, y_2\}$

$$y_h = C_1 y_1 + C_2 y_2$$

[2] Find y_{part} . (Use table)

[3] General Soln

$$y(x) = y_h + y_p = C_1 y_1 + C_2 y_2 + y_p$$

[Note]

to find constant in y_p Use y_{part} as Sol to non Homog D.E

Ex) Solve $y'' + -5y' + 4y = 8e^x \rightarrow f(x)$

[1] Homog: $y'' - 5y' + 4y = 0$

$$\text{char eqn} \rightarrow r^2 - 5r + 4 = 0 \Rightarrow \Delta = 25 - 4 \times 1 \times 4 = 9$$

$$r = 4, (r-4)(r-1) \rightarrow r_1$$

$$\text{F.S.S} \rightarrow \{e^{4x}, e^x\}$$

$$[2] \text{ Since } f(x) = 8e^x \rightarrow y_{part} = Ae^x$$

$$\text{as } e^x \text{ in F.S.S} \rightarrow y_{part} = Axe^x$$

Check 1

indep. Solns

$$y_p = Axe^x + Ae^x$$

$$y_p' = Axe^x + Ae^x + Ae^x = Axe^x + 2Ae^x$$

in D.E) $Axe^x + 2Ae^x - 5Axe^x - 5Ae^x + 4Axe^x = 8e^x$

$$2A - 5A = 8$$

$$-3A = 8 \Rightarrow A = -\frac{8}{3}$$

General Soln:

$$y(x) = C_1 e^{4x} + C_2 e^x + -\frac{8}{3} xe^x$$

21/11/2023

(19) 2.12

The Method of Undetermined Coeff.

$$ay'' + by' + cy = f(x) \rightarrow \begin{cases} e^{\alpha x + \beta} \\ P_n \\ \cos \theta / \sin \theta \end{cases}$$

general sol.

$$y(x) = y_h + y_p$$

$$\rightarrow y_h \rightarrow \text{F.S.S.} = \{y_1, y_2\}$$

$$y_h = C_1 y_1 + C_2 y_2$$

To find $y_{pw} \rightarrow$ ① by Table

② check (F.S.S)

I.C

$$y(x_0) = y_0, \quad y'(x_0) = y_1 \rightarrow \text{Find } C_1, C_2 \text{ General Sol.}$$

Ex) Find the form of y_{pw} (without constant)

$$\text{II } y'' + y = 8 \sin^2 x$$

$$* \text{Homog} \rightarrow y'' + y = 0$$

$$* \text{char. eqn} \rightarrow r^2 + 1 = 0 \rightarrow r = \pm i \Rightarrow r = \pm i$$

$$\alpha = 0 \\ \beta = 1$$

$$\text{F.S.S.} = \{\cos x, \sin x\}$$

$$y_h = C_1 \cos x + C_2 \sin x$$

$$p(x) = 8 \sin^2 x = 8 \left(\frac{1}{2} - \frac{1}{2} \cos 2x \right)$$

$$= \underbrace{4}_{P_{0(x)}} - \underbrace{4 \cos 2x}_{\cos \theta}$$

$$F_1(x) = 4 = P_{0(x)} \rightarrow y_{p1} = A \cdot 1$$

$$F_2(x) = 4 \cos 2x \rightarrow y_{p2} = B \cos 2x + C \sin 2x$$

$$y_p(x) = y_{p1} + y_{p2} = A + B \cos 2x + C \sin 2x$$

$$[2] \quad y'' + y' = \sinh(x)$$

$$\text{Homog} \Rightarrow y'' + y' = 0 \Rightarrow \text{char. eqn } r^2 - r = 0 \Rightarrow r(r-1) = 0$$

$$\begin{matrix} r=0 & r=1 \end{matrix}$$

$$\text{F.S.S} = \{1, e^x\}$$

$$f(x) = \sinh(x) = \frac{e^x - e^{-x}}{2} = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$$

$$f_1(x) = \frac{1}{2}e^x \rightarrow y_{p1}(x) = Ae^x \cdot x$$

$$f_2(x) = -\frac{1}{2}e^{-x} \rightarrow y_{p2}(x) = Be^{-x}$$

$$y_p(x) = Axe^x + Be^{-x}$$

$$[3] \quad y'' + y = \sin x \cos x$$

$$\text{Homog: } y'' + y = 0 \rightarrow r^2 + 1 = 0$$

$$\text{F.S.S} = \{\cos x, \sin x\}$$

$$f(x) = \frac{1}{2} \sin 2x$$

$$y_p(x) = A \cos 2x + B \sin 2x$$

Ex) If $y_{p0} = x^2$ is particular Sol. to $y'' - y = 2 - x$, $y(0) = 1$, $y'(0) = 0$, Find the general Sol.

$$\text{Homog: } y'' - y = 0 \leadsto \text{char. eqn } r^2 - 1 = 0 \Rightarrow (r-1)(r+1) = 0$$

$$\begin{matrix} r=1 & r=-1 \end{matrix}$$

$$y_h(x) = C_1 e^x + C_2 e^{-x}$$

$$\text{general Sol: } y(x) = C_1 e^x + C_2 e^{-x} + x^2$$

to Find C_1, C_2 Use I.C.

$$y(0) = 1 \leadsto C_1 + C_2 = 1 \quad \dots (1)$$

$$y'(x) = C_1 e^x - C_2 e^{-x} + 2x$$

$$y'(0) = 0 \leadsto C_1 - C_2 = 0 \leadsto C_1 = C_2 \quad \dots (2)$$

(2) ~~30/10/2020~~

$$C_1 + C_1 = 1 \leadsto 2C_1 = 1 \leadsto C_1 = \frac{1}{2} = C_2$$

$$\text{g.S} \Rightarrow \frac{1}{2}e^x + \frac{1}{2}e^{-x} + x^2 = \cosh(x) + x^2$$

Find the form of y_{pw}

① $y'' - y = x^2 e^{2x}$

$$y'' - y = 0 \Rightarrow r^2 - 1 = 0 \Rightarrow r_1 = 1, r_2 = -1$$

$$F.S.S = \{e^x, e^{-x}\}$$

$$F(x) = x^2 e^{2x} \Rightarrow y_{pw} = (a_2 x^2 + a_1 x + a_0) e^{2x} \\ = a_2 x^2 e^{2x} + a_1 x e^{2x} + a_0 e^{2x} \quad \#$$

② $y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}$

$$y'' - 6y' + 9y = 0$$

$$r^2 - 6r + 9 = 0$$

$$\Delta = 36 - 4 \times 1 \times 9 = 0$$

$$(r-3)^2 = 0 \Rightarrow \boxed{r=3} = r_1 = r_2$$

$$F.S.S = \{e^{3x}, x e^{3x}\}$$

$$F(x) = 6x^2 + 2 \Rightarrow y_{p1} = a_2 x^2 + a_1 x + a_0$$

$$F(x) = 12e^{3x} \Rightarrow y_{p2} = B e^{3x} \cdot x^2$$

$$y_{pw} = a_2 x^2 + a_1 x + a_0 + B x^2 e^{3x}$$

HW ③ $y'' + y = 4x + 10 \sin x$

$$y'' + y = 0 \Rightarrow r^2 + 1 = 0$$

$$F.S.S = \{\cos x, \sin x\}$$

$$F(x) = 4x \Rightarrow y_{p1} = a_1 x + a_0$$

$$F(x) = 10 \sin x \Rightarrow y_{p2} = (A \sin x + B \cos x) \cdot x \\ = Ax \sin x + Bx \cos x$$

$$\therefore y_{pw} = a_1 x + a_0 + Ax \sin x + Bx \cos x$$

Non Homog D.E

$$ay'' + by' + cy = f(x) \rightarrow \begin{cases} \text{EXP} \\ P_n \\ \sin \theta / \cos \theta \end{cases}$$

General soln $y(x) = y_h + y_p$

$$f.s.s = \{y_1, y_2\}$$

$$y_h = C_1 y_1 + C_2 y_2$$

To find y_p :-

① by table

② check!

* Solve / Find general Soln

① Find constant in y_p (consider y_p soln for Non-Homog. D.E)② Use I.C to find C_1 & C_2

Ex) Solve $y'' + 2y' + 2y = 5 \cos x + 5 \sin x \rightarrow y(0) = 1, y'(0) = 2$

Homog: $y'' + 2y' + 2y = 0$

char. eqn: $r^2 + 2r + 2 = 0$

$\Delta = 4 - 4 \times 1 \times 2 = -4 \rightarrow \text{complex root}$

$r = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm i \rightarrow \alpha = -1, \beta = 1$

f.s.s = $\{e^{-x} \cos x, e^{-x} \sin x\}$

$y_h = C_1 e^{-x} \cos x + C_2 e^{-x} \sin x$

$y_p(x) = A \sin x + B \cos x$

$y_p' = A \cos x - B \sin x$

$y_p'' = -A \sin x - B \cos x$

In. D.E

$-A \sin x - B \cos x + 2A \cos x + 2B \sin x + 2A \sin x + 2B \cos x = 5 \cos x + 5 \sin x$

$-A + 2B + 2A = 5 \quad \left\{ \begin{array}{l} -B + 2A + 2B = 5 \\ B + 2A = 5 \end{array} \right. \quad \left\{ \begin{array}{l} A + 2B = 5 \\ 5A = 15 \end{array} \right. \rightarrow A = 3$

$A + 2B = 5 \rightarrow 0$

$y_p = 3 \sin x - \cos x$

general Soln = $C_1 e^{-x} \cos x + C_2 e^{-x} \sin x + 3 \sin x - \cos x$

to find C_1 & C_2 use I.C $\Rightarrow C_1 = 2, C_2 = 1$

Ex) Find the form of $y_p(x)$

$$\Rightarrow y'' - 3y' - 4y = 3e^x + 2\sin x - 8e^x \cos 2x$$

$$\text{Homog: } y'' - 3y' - 4y = 0$$

$$\text{char. eqn: } r^2 - 3r - 4 = 0$$

$$\Delta = 9 - 4 \times (-4) = 25 \rightarrow r_1 = -1, r_2 = 4$$

$$\text{F.S.S.} = \{e^x, e^{4x}\}$$

$y_p(x)$ is-

$$y_{p1} = A e^x$$

$$y_{p2} = B \sin x + C \cos x$$

$$y_{p3} = (D \cos 2x + E \sin 2x) e^x$$

$$y_p(x) = A e^x + B \sin x + C \cos x + D e^x \cos 2x + E e^x \sin 2x$$

General Sol for non-Homog. D.E

$$y(x) = y_h + y_p$$

$$F.S.S. = \{y_1, y_2\} \rightarrow y_h = C_1 y_1 + C_2 y_2$$

* undetermined coeff.

II by Table (Su) $\begin{cases} \rightarrow \exp \\ \rightarrow P_n \cos / \sin \end{cases}$

III check! (compare by F.S.S)

* undetermined coefficient & The form of $y_p(x)$ where Su is product

Rule 1 Product of (Poly. exp)

$$\text{For the D.E } ay'' + by' + cy = CX^m e^{rx}$$

where $m \geq 0$ is positive integer, $m = 1, 2, 3, \dots$

OR X^m is Poly of degree(m)

Then the form of y_p is

$$y_{px} = X^S (A_m X^m + A_{m-1} X^{m-1} + \dots + A_1 X + A_0) e^{rx}$$

where S is the number of repeated root r in char. eqn

$S=0 \rightarrow$ if r is not root in char. eqn

$S=1$ / $S=2 \rightarrow$ (repeated root x)

Ex Find the form of y_{px} to D.E $y'' + 2y' - 3y = 2(x)$

$$a) L(x) = 3Xe^x$$

$$\pm \text{Homog} \rightarrow y' + 2y' - 3y = 0 \Rightarrow \text{char. eqn} \rightarrow r^2 + 2r - 3 = 0$$

$$\begin{matrix} m=1 \\ r=1 \\ S=1 \end{matrix}$$

$$\begin{matrix} (r+1)(r-3) = 0 \\ \underline{r=1} \text{ or } \underline{r=3} \end{matrix}$$

$$y_{px} = X^1 (A_1 X + A_0) e^x$$

$$b) L(x) = 5x^2 e^{-2x} \rightarrow m=2, r=-2, S=0$$

$$\downarrow y_{px} = X^0 (A_2 X^2 + A_1 X + A_0) e^{-2x}$$

$\downarrow$ AS $r=-2$ not root in char. eqn

Rule 2

$$\text{If } f(x) = Cx^m e^{\alpha x} \cos(\beta x)$$

$$\text{OR } f(x) = Cx^m e^{\alpha x} \sin(\beta x)$$

Then,

$$y_{\text{part}} = x^S (A_m x^m + A_{m-1} x^{m-1} + \dots + A_1 x^1 + A_0) e^{\alpha x} \cos(\beta x) \\ + x^S (B_m x^m + \dots + B_1 x^1 + B_0) e^{\alpha x} \sin(\beta x)$$

where S is the number of repeated complex root

$r = \alpha \pm i\beta$ in char. eqn

$S=0 \rightarrow$ is no complex root in char.

$S=1 \rightarrow$ is $r = \alpha \pm i\beta$ in char. eqn.

Ex. Find the form of y_{part} to D.E

$$\text{I] } y'' + 2y' + 2y = 7x e^{-x} \cos x.$$

$$\text{Homog} \rightarrow y'' + 2y' + 2y = 0$$

$$\text{char. eqn} \rightarrow r^2 + 2r + 2 = 0$$

$$\Delta = 4 - 4 \times 1 \times 2 = -4 < 0$$

$$r = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm i \rightarrow \alpha = -1, \beta = 1$$

$$f(x) = 7x e^{-x} \cos x \rightarrow m=1, \alpha=-1, \beta=1$$

$$y_{\text{part}} = x^S (A_1 x + A_0) e^{-x} \cos x \\ + x^S (B_1 x + B_0) e^{-x} \sin x. \quad \left. \begin{array}{l} r = -1 \pm i \\ S=1 \end{array} \right\}$$

$$\text{[2] } y'' + 2y' - 3y = x e^{2x} \sin(3x) \rightsquigarrow m=1 / \alpha=2 / \beta=3 / r=2 \pm i\beta \\ S=0$$

$$\text{Homog} \rightarrow y'' + 2y' - 3y = 0$$

$$\text{char eqn} \rightarrow r^2 + 2r - 3 = 0$$

$$\rightsquigarrow \Delta = 4 - 4 \times -3 \times 1 = 16 > 0$$

$$r = -3, (r+3)(r-1) \rightarrow r=1$$

$$y_{\text{part}} = x^0 (A_1 x + A_0) \cos(3x) e^{2x} + x^0 (B_1 x + B_0) e^{2x} \sin 3x$$

* Superposition Principle :

$$\text{Let } L[y] = ay'' + by' + cy$$

$$\text{If } y_1 \text{ be solution to } L[y] = f(x)$$

$$\text{If } y_2 \text{ be solution to } L[y] = g(x)$$

$$\text{Then, Sol for D.E } L[y] = C_1 f(x) + C_2 g(x).$$

$$\text{is } C_1 y_1 + C_2 y_2$$

$$\text{Ex) let } L[y] = y'' + p(x)y' + q(x)y$$

$$\text{If } y_1 = \cos x \text{ is sol to D.E } L[y] = 2 \sin x$$

$$y_2 = e^{2x} \text{ is sol to D.E } L[y] = \frac{e^{2x}}{3}$$

$$\text{Find Sol to D.E } L[y] = 4 \sin x + \frac{5}{3} e^{2x}$$

$$\text{is } C_1 y_1 + C_2 y_2 = C_1 \cos x + C_2 e^{2x}$$

$$C_1 f(x) + C_2 g(x) = C_1 \sin x + 5 e^{2x}$$

$$C_1 \cdot 2 \sin x + \frac{C_2 e^{2x}}{3} = 4 \sin x + 5 e^{2x}$$

$$2C_1 = 4 \rightarrow C_1 = 2$$

$$\frac{C_2}{3} = 5 \rightarrow C_2 = 15$$

$$\Rightarrow 2 \cos x + 15 e^{2x}$$

$$\text{(H.w)} \quad y_1 \text{ Sol } y'' + x^2 y' + y = x^2 - 2x$$

$$y_2 \quad L[y] = 5x + 3$$

$$\text{Find Sol } y'' + x^2 y' + y = 2x^2 + 11x + 9.$$

Non-Homog D.E

$$ay'' + by' + cy = S(x)$$

Use undetermined coeff. iff

Sol: Pm or e^{x+ib} or $\sin$ or $\cos$

Rule

$$f(x) = cx^m e^{rx}$$

$$S(x) = c x^m e^{i\alpha x} \cos \beta x \quad \text{or} \quad cx^m e^{i\alpha x} \sin \beta x$$

Ex $S(x) = \frac{x}{x^2 + 2x}$, $S(x) = \tan x$ / $\sec x$

und etc. (etc. etc. etc.)

Variation of Parameter Method &

Standard form: $y'' + P(x)y' + Q(x)y = g(x)$

General Sol: $y_{inh} = y_h + y_p = C_1 y_1 + C_2 y_2 + y_p$

* No special case for $g(x)$ (Non-homog. part)** By this method the $y_p(x)$ has No constantSteps: Solve $Ly'' + P(x)y' + Q(x)y = g(x)$

① Find F.S.S = $\{y_1, y_2\} \Rightarrow y_h = C_1 y_1 + C_2 y_2$

② Find $W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$

③ Find $y_p(x) = y_1 u_1 + y_2 u_2$

As y_p satisfy Non-Homog D.E $\Rightarrow$ Find u_1, u_2

$$u_1 = -\int \frac{y_2 g(x)}{W(x)} dx \quad / \quad u_2 = \int \frac{y_1 g(x)}{W(x)} dx$$

Ex) Let $L[y] = y'' + p(x)y' + q(x)y$. If F.S.S. = $\{x, x^2\}$

for $W(y) \neq 0$. Use Variation to find $y_p(x)$?

For D.E $L[y] = x$?

① ~~also~~ $\rightarrow \{x, x^2\}$

$$\textcircled{2} W[x, x^2] = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2$$

$$* y'' + p(x)y' + q(x)y = \underline{x} \rightarrow q(x)$$

$$\textcircled{3} * U_1 = - \int \frac{x^2 \cdot x}{x^2} dx = - \int x dx = -\frac{x^2}{2}$$

$$* U_2 = \int \frac{x \cdot x}{x^2} dx = \int 1 dx = x$$

$$y_p = x + \frac{-x^2}{2} + x^2 \cdot x = -\frac{x^3}{2} + x^3 = \frac{x^3}{2}$$

General Sol & $y_{hom} = C_1 x + C_2 x^2 + \frac{x^3}{2}$

* Solve $y'' - 2y' + y = \frac{e^x}{1+x^2}$, $y(0) = 1$, $y'(0) = 1$

① Homog $\rightarrow y'' - 2y' + y = 0 \rightarrow$ char. eq. $= r^2 - 2r + 1 = 0 \rightarrow \Delta = 4 - 4 \cdot 1 \cdot 1 = 0$
 F.S.S. = $\{e^x, xe^x\}$ $\hookrightarrow (r-1)^2 = 0 \rightarrow \boxed{r=1}$

② $W[e^x, xe^x] = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = xe^{2x} + e^{2x} - e^{2x} = e^{2x}$

* $g(x) = \frac{e^x}{1+x^2}$

③ * $U_2 = - \int \frac{e^x \cdot e^x}{\frac{e^{2x}}{1+x^2}} dx = - \int \frac{1}{1+x^2} dx = -\tan^{-1} x$

* $U_1 = \int \frac{xe^x \cdot e^x}{\frac{e^{2x}}{1+x^2}} dx = \frac{1}{2} \int \frac{2x}{1+x^2} dx = \frac{1}{2} \ln|1+x^2|$

~~General Sol~~ $y_p(x) = \frac{1}{2} e^x \ln|1+x^2| - xe^x \tan^{-1} x$

General Sol: $C_1 e^x + C_2 xe^x + y_p$

* Use I.C to Find C_1, C_2

Solve 8 $x^2 y'' - 2xy' + 2y = x^3 e^{-x} \rightarrow a=1, b=-2, c=2, b-a=-3$
 $\hookrightarrow$ Cauchy Euler, second order, Non-Homog.

① Homog $\rightarrow x^2 y'' - 2xy' + 2y = 0 \rightarrow$ char. eqn $r^2 - 3r + 2 = 0$
 $\Delta \rightarrow 9 - 4 \cdot 1 \cdot 2 = 1 > 0 \rightarrow (r-2)(r-1) = 0 \rightarrow r=1, r=2$

● L.S.S. = $\{x^1, x^2\}$

② $W[E_1, E_2] = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2$

③ $y'' - \frac{2}{x}y' + \frac{2}{x^2}y = \frac{x e^{-x}}{x^2} \quad g(x)$

$$u_1 = - \int \frac{x^2 \cdot x e^{-x}}{x^2} dx = - \int x e^{-x} dx = -(-x e^{-x} - e^{-x}) = x e^{-x} + e^{-x}$$

$$u_2 = \int \frac{x e^{-x} \cdot x}{x^2} dx = \int e^{-x} dx = -e^{-x}$$

$$y_p(x) = x(x e^{-x} + e^{-x}) + x^2(-e^{-x}) = x^2 e^{-x} + x e^{-x} - x^2 e^{-x} = x e^{-x}$$

general Sol. 8 $y(x) = C_1 x + C_2 x^2 + x e^{-x}$

* Find the DE if the general Sol. is 8:

$$y(x) = C_1 x + C_2 x \ln x + x (\ln x)^2$$

order = 2, Cauchy Euler, Non-Homog.

$$ax^2 y'' + bxy' + cy = g(x)$$

$y_1 = x, y_2 = x \ln x \rightarrow$ Repeated root

$r_1 = r_2 = 1 \rightarrow$ char. eq. $r(r-1)^2 = 0 \Rightarrow r^2 - 2r + 1 = 0$

Is 9 * $a=1, b-a=-2, c=1 \Rightarrow b=-1$

$$x^2 y'' - xy' + y = g(x)$$

* As $y_p = x (\ln x)^2$ satisfy Non-Homog. D.E. So 8

$$x^2 y_p'' + x y_p' + y_p = \boxed{g(x)}$$

H.W Solve 8 $y'' + y = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$

20-11-2023

(23) 5/13

* Variation of Parameter

$$y'' + P(x)y' + Q(x)y = g(x)$$

$$y_p = y_1 u_1 + y_2 u_2$$

← Q * Use the method of undetermined coeff. to find $y_p(x)$
for D.E $(x^2 y'' + xy' - 16y = 10x^4 \ln x), x > 0$

* Non-Homog. Cauchy-Euler

$$ax^2 y'' + bxy' + cy = g(x), x > 0$$

Let $x = e^t \rightarrow t = \ln x \rightarrow$ Transform Cauchy to constant coeff

$$y' = \frac{dy}{dx} \xrightarrow{\text{dep}} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \left(-\frac{1}{x}\right)$$

$$y'' = \frac{1}{x} \left(\frac{d^2 y}{dt^2} \cdot \frac{dt}{dx} \right) + \frac{dy}{dt} \cdot \frac{1}{x^2}$$

$$= \frac{1}{x^2} \frac{d^2 y}{dt^2} - \frac{1}{x^2} \frac{dy}{dt}$$

$$\text{in D.E} \rightarrow ax^2 \left(\frac{1}{x^2} \frac{d^2 y}{dt^2} - \frac{1}{x^2} \frac{dy}{dt} \right) + b \left(\frac{1}{x} \frac{dy}{dt} \right) + cy = G(t)$$

$$a \frac{d^2 y}{dt^2} - a \frac{dy}{dt} + b \frac{dy}{dt} + cy = G(t)$$

$$(H) \text{ indep } \rightarrow a y'' + (b-a) y' + cy = G(t)$$

* Solve constant coeff To find $y(t)$

Then use $t = \ln x$

To find Sol in $y(x)$.

Q] $x^2 y'' + xy' - 16y = 10x^4 \ln x$

$a=1, b=1, a-b=0, c=-16$

let $x=e^t \rightarrow t=\ln x$

constant coeff. & $a \frac{d^2 y}{dt^2} + (b-a) \frac{dy}{dt} + cy = G(t)$

$\hookrightarrow y'' - 16y = 10te^{4t}$

* Homog: $y'' - 16y = 0 \rightarrow$ char. eqn: $r^2 - 16 = 0 \rightarrow (r-4)(r+4) = 0$

F.S.S. = $\{e^{4t}, e^{-4t}\}$

$\begin{matrix} \uparrow & \uparrow \\ r=4 & r=-4 \end{matrix}$

$\hookrightarrow y_h = C_1 e^{4t} + C_2 e^{-4t}$

* $G(t) = 10te^{4t} \rightarrow m=1, r=4, S=1$

$\hookrightarrow y_p(t) = t^1(a_1 t + a_0)e^{4t}$
 $= (a_1 t^2 + a_0 t)e^{4t}$

to find $y_p(x)$ use $t=\ln x$

$\hookrightarrow (a_1 (\ln x)^2 + a_0 \ln x)x^4$

Q] Find general Sol.

$(xy')' + \frac{1}{x}y = \frac{\sec(\ln x)}{x}$

$\hookrightarrow (x^2 y'' + xy' + y = \frac{\sec(\ln x)}{x}) \times x$

$\hookrightarrow x^2 y'' + xy' + y = \sec(\ln x)$

$a=1, b=1, c=1, b-a=0$ and let $K=e^t \rightarrow t=\ln x$

sl indep(t) $\hookrightarrow y'' + y = \sec(t)$

Solve by variation of parameter

① Homog $\Rightarrow y'' + y = 0 \rightarrow$ char. eqn: $r^2 + 1 = 0 \rightarrow r = \pm i \rightarrow \alpha=0, \beta=1$

F.S.S. = $\{\cos(t), \sin(t)\}$

② $w[\cos t, \sin t] = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$

③ $u_1 = -\int \sin t \cdot \sec t dt = \ln |\cos t|$

$u_2 = \int \cos t \cdot \sec t dt = t$

$y_p t = \cos t \cdot \ln |\cos t| + t \sin t$

$y_p(x) = \cos(\ln x) \cdot \ln |\cos(\ln x)|$

$+ (\ln x) \sin(\ln x)$

General Soln:

$$y(x) = C_1 \cos(\ln x) + C_2 \sin(\ln x) + \cos(\ln x) \cdot \ln \cos(\ln x) + \ln x \sin \ln x$$

H.W) ① $x^2 y'' + x y' + 9y = \sec(3 \ln x)$

② $x^2 y'' - x y' + y = \ln x$

CH 6 Higher order with constant coefficient

Homog:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

$y^{(n)} \rightarrow r^n$

* char. eqn:

$$a_n r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0 = 0$$

Case 1

If $r_1 \neq r_2 \neq \dots \neq r_n$

$$\hookrightarrow \text{f.s.s} = \{e^{r_1 x}, e^{r_2 x}, \dots, e^{r_n x}\}$$

Case 2

If r root for char. eqn and repeated k -times

$$r_1 = r_2 = \dots = r_k = r$$

$$\hookrightarrow \text{F.S.S} = \{e^{rx}, x e^{rx}, x^2 e^{rx}, \dots, x^{k-1} e^{rx}\}$$

Case 3

Complex root

$$r_{1,2} = \alpha \pm i\beta \quad / \quad r_{3,4} = \alpha \pm i\beta$$

$$\hookrightarrow \text{F.S.S} = \{e^{\alpha x} \cos(\beta x), e^{\alpha x} \sin(\beta x), e^{\alpha x} \cos(\beta x), e^{\alpha x} \sin(\beta x)\}$$

Ex) Solve

$$① y^{(5)} + 2y^{(4)} + y^{(3)} = 0$$

↳ order = 5, constant coeff, Homog.

$$\text{Char. eqn } r^5 + 2r^4 + r^3 = 0$$

$$r^3(r^2 + 2r + 1) = 0$$

$$r_1 = r_2 = r_3 = 0 \quad \hookrightarrow r_4 = r_5 = -1$$

$$\text{F.S.S.} = \{e^{0x}, xe^{0x}, x^2e^{0x}, e^{-x}, xe^{-x}\}$$

$$= \{1, x, x^2, e^{-x}, xe^{-x}\}$$

$$\text{General Soln } y(x) = C_1 + C_2x + C_3x^2 + C_4e^{-x} + C_5xe^{-x}$$

$$② y''' + 3y'' - 4y = 0$$

↳ order = 3, Homog, Cons. Coeffs

$$\text{Char. eqn } r^3 + 3r^2 - 4 = 0$$

$$1 + 3 - 4 = 0 \quad \checkmark \quad ① \text{ by } r=1$$

$$\text{Factor } (r-1) = 0$$

$$(r-1)(r^2 + 4r + 4) = 0$$

$$r = 1 \quad \hookrightarrow r = -2 = r_2 = r_3$$

$$4 \quad \hookrightarrow r_1, r_2, r_3$$

$$\begin{array}{r} r^2 + 4r + 4 \\ r=1 \overline{) r^3 + 3r^2 - 4} \\ \underline{r^3 - r^2} \quad \ominus \\ 4r^2 - 4 \\ \underline{4r^2 - 4r} \quad \ominus \\ 4r + 4 \\ \underline{4r + 4} \quad \ominus \\ 0 \end{array}$$

$$\text{F.S.S.} = \{e^x, e^{-2x}, xe^{-2x}\}$$

$$y(x) = C_1e^x + C_2e^{-2x} + C_3xe^{-2x}$$

Homog Higher order D.E

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

$$\text{Char. eqn: } y^{(n)} \rightarrow r^n \text{ i.e. } (y \rightarrow 1)$$

Ex] Solve

$$\textcircled{3} y'' + 3y' - 10y = 0$$

$$\text{Char. eqn: } r^2 + 3r - 10 = 0 \rightarrow r(r+5) - 2(r-2) = 0$$

$$\begin{matrix} r=0 & r=2 & r=-5 \end{matrix}$$

$$\text{F.S.S.} = \{1, e^{2x}, e^{-5x}\}$$

$$\textcircled{4} \frac{d^4 y}{dx^4} + 2 \frac{d^2 y}{dx^2} + y = 0$$

$$\text{Char. eqn: } r^4 + 2r^2 + 1 = 0 \rightarrow (r^2 + 1)^2 = 0$$

$$\begin{matrix} r_{1,2} = \pm i \\ r_{3,4} = \pm i \end{matrix}$$

$$\text{F.S.S.} = \{\cos x, \sin x, x \cos x, x \sin x\}$$

Type 2 : Find D.E

Ex] If the root for char. eqn are $r_1 = 4$, $r_2 = r_3 = -5$ find D.E

$$* r_1 = 4 \xrightarrow{\text{Factor}} (r-4) = 0$$

$$* r_2, r_3 = -5 \rightarrow (r+5)^2 = 0$$

$$\text{Char. eqn: } (r-4)(r+5)^2 = 0 \rightarrow (r-4)(r^2 + 10r + 25) = 0$$

$$\rightarrow r^3 + 6r^2 - 15r - 100 = 0$$

$$* \text{Use } r^n \rightarrow y^{(n)}$$

$$\rightarrow y''' + 6y'' - 15y' - 100y = 0 \quad \#$$

Ex2) If root for char. eqn are

$$r_1 = -\frac{1}{2}, r_2, r_3 = 3 \pm i$$

Find D.E.

$$* r = -\frac{1}{2} \rightarrow (r + \frac{1}{2}) = 0$$

$$* r = 3 + i \rightarrow (r - 3) = i \rightarrow (r - 3)^2 = -1 \rightarrow r^2 - 6r + 9 = -1$$

$$\hookrightarrow r^2 - 6r + 10 = 0 \text{ * Factor}$$

$$* \text{char. eqn } (r + \frac{1}{2})(r^2 - 6r + 10) = 0$$

$$\hookrightarrow r^3 - 6r^2 + 10r + \frac{r^2}{2} - 3r + 5 = 0$$

$$\text{char. eqn } \hookrightarrow r^3 - 5.5r^2 + 7r + 5 = 0$$

$$\text{D.E. } \hookrightarrow y''' - 5.5y'' + 7y' + 5y = 0$$

Q1) Find the Homog D.E when the general sol is

$$y(x) = 7x^2 + 9xe^{3x} - \sin(2x)$$

A) Find the order. (7)

B) Find D.E.

$\hookrightarrow r=3$
repeated
twice
 $r_3 = r_4 = 3$

$\hookrightarrow$ complex root $a=0$
 $b=2$

$$\hookrightarrow x^{k-1} = x^2 \rightarrow k-1 = 2 \rightarrow k=3 \text{ repeated 3-times}$$

$$* 7x^2 \rightarrow r_1 = r_2 = r_3 = 0 \rightarrow r^3 = 0$$

$$* 9xe^{3x} \rightarrow r_4 = r_5 = 3 \rightarrow (r-3)^2 = 0$$

$$* -\sin(2x) \rightarrow r_6, 7 = \pm 2i \rightarrow r^2 = 4 \rightarrow r^2 + 4 = 0$$

$$B) r^3 * (r-3)^2 * (r^2+4) = 0 \rightarrow \text{char. eqn}$$

$$\hookrightarrow (r^3 + 4r^3)(r^2 - 6r + 9) = 0$$

$$\hookrightarrow r^7 - 6r^6 + 9r^5 + 4r^5 - 24r^4 + 36r^3 = 0$$

$$* \text{Use } r^n \rightarrow y^{(n)}$$

$$\hookrightarrow y^{(7)} - 6y^{(6)} + 9y^{(5)} + 4y^{(5)} - 24y^{(4)} + 36y^{(3)} = 0 \quad \#$$

Non-Homog Higher order.

↳ By Using Undetermined coeff.

Ex Find the form of y_{part} .

$$\textcircled{1} y''' + y'' - 2y = xe^x$$

$$\text{Homog} \rightarrow y'' + y' - 2y = 0$$

$$\text{char. eqn} \rightarrow r^3 + r^2 - 2 = 0$$

$$r=1 \rightarrow 1+1-2=0$$

$$r=1 \quad (r-1)(r^2+2r+2)=0 \rightarrow \Delta = 4-4+2 = -4 \quad / \quad r_{2,3} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$$\text{F.S.S.} = \{e^x, e^x \cos(x), e^x \sin(x)\}$$

$$* f(x) = xe^x \rightarrow m=1, r=1, S=1$$

$$y_{\text{part}} = x'(a_1x + a_0)e^x$$

$$= (a_1x^2 + a_0x)e^x = a_1x^2e^x + a_0xe^x$$

$$\textcircled{2} y''' + 9y' = x \sin(3x) + x^2 e^{2x}$$

$$\text{Homog: } y''' + 9y' = 0$$

$$\text{char. eqn: } r^3 + 9r = 0 \rightarrow r(r^2 + 9) = 0$$

$$r=0 \quad r_{2,3} = 0 \pm 3i$$

$$\text{F.S.S.} = \{1, \cos(3x), \sin(3x)\}$$

$$* f_1(x) = x \sin(3x) e^{0x} \rightarrow (m=1, r=\pm 3i, S=1)$$

$$\text{or } \alpha=0, \beta=3$$

$$\rightarrow y_{p1}(x) = x(a_1x + a_0) \sin(3x) + x(b_1x + b_0) \cos(3x)$$

$$* f_2(x) = x^2 e^{2x} \rightarrow (m=2, r=2, S=0)$$

$$\rightarrow y_{p2}(x) = x^2(a_2x^2 + a_1x + a_0)e^{2x}$$

$$* y_p = y_{p1} + y_{p2} = (a_1x^2 + a_0) \sin(3x) + (b_1x^2 + b_0x) \cos(3x) + (a_2x^2 + a_1x + a_0) e^{2x}$$

$$\textcircled{3} \quad y^{(4)} + 2y'' + y = 2 \sin x - 5 \cos x$$

$$\text{Char. eqn} \rightarrow r^4 + 2r^2 + 1 = 0$$

$$(r^2 + 1)^2 = 0$$

$$\hookrightarrow r_{1,2} = \pm i, \quad r_{3,4} = \pm i$$

$$\text{F.S.S.} = \{ \cos x, \sin x, x \cos x, x \sin x \}$$

$$\text{By rule } y_p(x) = (A \cos x + B \sin x) x^2$$

$$* \text{ By rules } f(x) = 2 \sin x - 5 \cos x \rightarrow \alpha = 0, \beta = 1$$

$$m = 0, \quad r = \pm i, \quad S = 2$$

$$y_p(x) = x^2 (A \cos x + B \sin x)$$

$$\textcircled{4} \text{ H.W. 8 } y^{(4)} + y^{(3)} = 1 + e^{-x}$$

Non-Homog Higher order DE

Ex) Find the form of $y_{p(x)}$ for the following D.E

$$① y^{(4)} + y^{(3)} = 1 + e^{-x}$$

$$\text{Homog: } y^{(4)} + y^{(3)} = 0$$

$$\text{char. eqn: } r^4 + r^3 = 0$$

$$r^3(r+1) = 0$$

$$r^3 = 0 \Rightarrow r_1 = r_2 = r_3 = 0 \quad r_4 = -1$$

To find $y_{p(x)}$:

$$* f_1(x) = 1 \cdot x^0 \cdot e^x$$

$$r=0, m=0, S=3 \Rightarrow y_{p1}(x) = Ax^3$$

$$* f_2(x) = e^{-x}$$

$$r=-1, m=0, S=1 \Rightarrow y_{p2}(x) = x(A)e^{-x}$$

$$* y_{p(x)} = y_{p1} + y_{p2} = Ax^3 + Bxe^{-x}$$

Ex) Find the form of $y_{p(x)}$ for D.E
 $\hookrightarrow (y^{(4)} + 2y''' + 5y'' = f(x))$

where:

$$a) f(x) = 4x^2$$

$$\text{Homog: } y^{(4)} + 2y''' + 5y'' = 0$$

$$\text{char. eqn: } r^4 + 2r^3 + 5r^2 = 0$$

$$r^2(r^2 + 2r + 5) = 0$$

$$r^2 = 0 \Rightarrow r_1 = r_2 = 0$$

$$\rightarrow r = -2 \pm 4i = -1 \pm 2i$$

$$* f(x) = 4x^2 \Rightarrow m=2, r=0, S=2$$

$$\hookrightarrow y_{p(x)} = x^2(a_2x^2 + a_1x + a_0)$$

$$\begin{aligned} \text{b) } f(x) &= 4 \cos^2 x + 4 \sin^2 x \\ &= 4(\cos^2 x + \sin^2 x) \quad (1) \\ &= 4+1 = \boxed{4} \end{aligned}$$

$$m=0, r=0, S=2$$

$$y_p(x) = x^2(A)$$

$$\text{c) } f(x) = x \sin(2x) e^{0x}$$

$$m=1, r=0 \pm 2i, S=0$$

$$y_p(x) = x^0 (a_1 x + a_0) \sin(2x) + x^0 (Bx + B_0) \cos(2x)$$

$$\text{* Q.1) If } y = \overbrace{C_1 + C_2 x}^{y_h} + \overbrace{C_3 e^x + 2x^3}^{y_p}$$

In general sol for D.E $L[y] = g(x)$.

1) Find order. 3

2) Find Type. Non-Homog / constant coeff

3) Find $g(x)$.

$$\text{D.E} \rightarrow \text{F.S.S} = \{1, x, e^x\}$$

$$\begin{aligned} \text{Let for char. eqn} \quad & \left[\begin{aligned} r_1 = r_2 = 0 & \rightarrow r^2 = 0 \\ r_3 = 1 & \rightarrow r-1 = 0 \end{aligned} \right. \end{aligned}$$

$$\text{Char. eqn} \& \; r^2(r-1) = 0$$

$$r^3 - r^2 = 0$$

$$(\text{Homog D.E}) \quad r^n \rightarrow y^{(n)}$$

$$y^{(3)} - y'' = 0$$

$$(\text{Non-Homog. D.E}) \rightarrow y''' - y'' = g(x)$$

Consider $y_p = 2x^3$ as Sol. for non Homog D.E

$$(y_p' = 6x^2), (y_p'' = 12x), (y_p''' = 12)$$

$$g(x) = 12 - 12x$$

D.E &

$$y''' - y'' = 12 - 12x$$

Q) Given that $(r+7)(r^2-1)(r^2+1) = 0$
 is an auxiliary eqn for D.E $Ly = 0$
 Find the Sol to D.E $Ly = e^{-7x} \cos x \rightarrow \alpha = -7, \beta = 1$

$$\hookrightarrow r = -7 \pm i$$

$$(r_1 = -7) (r_2 = r_3 = 1) (r_4 = r_5 = -1) (r_6 = 7 \pm i)$$

$$F.S.S = \{ e^{-7x}, e^x, xe^x, e^{-x}, xe^{-x}, \cos x, \sin x \}$$

$$y_h = C_1 \downarrow + C_2 \downarrow + \dots + C_7 \sin x$$

$$y_p(x) = A e^{-7x} \cos x + B e^{-7x} \sin x$$

$$* r = -7 \pm i$$

$$y(x) = y_h + y_p$$

$$m=0, S=0$$

Ex) Find the form of $y_p(x)$ for D.E

$$y^{(6)} + 3y^{(5)} + 4y^{(4)} + 2y^{(3)} = x^2 + \sin(2x)$$

$$\text{Char. eqn } r^6 + 3r^5 + 4r^4 + 2r^3 = 0$$

$$r^3(r^3 + 3r^2 + 4r + 2) = 0$$

$$r = -1 \rightarrow -1 + 3 - 4 + 2 = 0 \checkmark$$

$$r^3(r+1)(r^2+2r+2) = 0$$

$$r_1 = r_2 = r_3 = 0 \quad \& \quad \hookrightarrow r_4 = -1 \quad \hookrightarrow r_{5,6} = -2 \pm 2i$$

$$\hookrightarrow = -1 \pm i$$

$$\begin{array}{r} r^2 + 2r + 2 \\ r+1 \overline{) r^3 + 3r^2 + 4r + 2} \\ \underline{r^2 + r^2} \ominus \\ 2r^2 + 4r + 2 \\ \underline{2r^2 + 2r} \ominus \\ 2r + 2 \\ \underline{2r + 2} \ominus \\ 0 \end{array}$$

$$* S_1(x) = x^2 \rightarrow m=2, r=0, S=3$$

$$\rightarrow y_{p1}(x) = x^3 (A_2 x^2 + A_1 x + A_0)$$

$$* S_2(x) = \sin(2x) \rightarrow m=0, r = 0 \pm 2i, S=0$$

$$\rightarrow y_{p2}(x) = A \cos(2x) + B \sin(2x)$$

$$* y_p = y_{p1} + y_{p2} = x^3(A_2 x^2 + A_1 x + A_0) + A \cos(2x) + B \sin(2x)$$

Chapter (7) : Laplace Transformation

$$f(t) \xrightarrow{\mathcal{L}} F(s)$$

$$\xleftarrow{\mathcal{L}^{-1}}$$

Defn: If $f(t)$ is cont on $[0, \infty)$ then

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} f(t) e^{-st} dt, \quad s > 0$$

$$= \lim_{b \rightarrow \infty} \int_0^b f(t) e^{-st} dt$$

Recall that : If limit exist, then Improper Integral cont.
 = If limit doesn't exist $\rightarrow$ div

discont infinite $\rightarrow$

all $f(t)$ & t are all $f(t) \rightarrow \infty$ as $t \rightarrow \infty$

discont finite $\rightarrow$

all $f(t)$ are $\rightarrow \infty$ as $t \rightarrow \infty$

Defn: $f(t)$ is called piecewise cont

on $I = [a, b]$. If $f(t)$ is cont at

all points in I , except finite number of discont

Ex] If $f(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 2 & 1 \leq t < 2 \\ (t-2)^2 & 2 \leq t \leq 3 \end{cases} \rightarrow f(t) \text{ is piecewise cont}$
 (discont. pts. $t=1, 2$)

properties:

1) Linearity $\mathcal{L}[cf(t) + g(t)] = \mathcal{L}[cf(t)] + \mathcal{L}[g(t)] = cF(s) + G(s)$

2) $\mathcal{L}[cf(t)] = c \mathcal{L}[f(t)] = cF(s)$

Ex, Evaluate :

1) $\mathcal{L}[c] = \int_0^{\infty} c \cdot e^{-st} dt = \lim_{b \rightarrow \infty} \int_0^b c \cdot e^{-st} dt = \lim_{b \rightarrow \infty} \left(c \cdot \frac{e^{-st}}{-s} \right) = -\frac{c}{s} \left(e^{-sb} - 1 \right)$

$\Rightarrow -\frac{c}{s} (0 - 1) = \boxed{\frac{c}{s}}$ * Rule $\mathcal{L}[c] = \frac{c}{s}$ *

② $\mathcal{L}[e^{at}]$, where $a \in \mathbb{R}$

$$\mathcal{L}[e^{at}] = \int_0^{\infty} e^{at} \cdot e^{-st} dt.$$

$$= \lim_{b \rightarrow \infty} \int_0^b e^{at} \cdot e^{-st} dt = \lim_{b \rightarrow \infty} \int_0^b e^{t(a-s)} dt = \lim_{b \rightarrow \infty} \left[\frac{e^{(a-s)t}}{a-s} \right]_0^b$$

$$\lim_{b \rightarrow \infty} \frac{1}{a-s} (e^{(a-s)b} - 1)$$

* Two cases *

① If $a-s < 0 \rightarrow a < s$

② If $s-a < 0$

① If $a-s < 0 \rightarrow a < s$

$$\mathcal{L}[e^{at}] = -\frac{1}{a-s} = \frac{1}{s-a}$$

② $a-s > 0 \rightarrow \text{div}$

Ex $\mathcal{L}[e^{-5t}] \rightarrow \mathcal{L}[e^{-5t}] = \frac{1}{s+5}$

* Tabel of Laplace Transform.

$\mathcal{L}(f)$	$\mathcal{L}[\mathcal{L}(f)] = F(s)$
① C	$\frac{C}{s}$
② e^{at}	$\frac{1}{s-a}$
③ $\sin(at)$	$\frac{a}{s^2+a^2}$
④ $\cos(at)$	$\frac{s}{s^2+a^2}$
⑤ $\sinh(at)$	$\frac{a}{s^2-a^2}$
⑥ $\cosh(at)$	$\frac{s}{s^2-a^2}$
⑦ t^n	$\frac{n!}{s^{n+1}}$

Ex * $\lim_{s \rightarrow \infty} F(s) = 0 \Rightarrow \lim_{s \rightarrow \infty} \frac{C}{s} = 0, \lim_{s \rightarrow \infty} \frac{s}{s^2+a^2} = 0$

Ex Find the value of K such that, $F(s) = \frac{2KS}{s+1} - \frac{s^3-1}{s^3+1}$

* As $\lim_{s \rightarrow \infty} F(s) = 0 \Rightarrow \lim_{s \rightarrow \infty} \frac{2KS}{s+1} = 0 + \lim_{s \rightarrow \infty} \frac{s^3-1}{s^3+1} = 0$

$2K-1=0 \Rightarrow K = \frac{1}{2}$

(How) Find the value of a, b .

$$F(s) = \frac{as}{s+1} - \frac{e^{bs}}{s+1} - \frac{s^2-1}{s^2+1}$$

12-12-2023

(27) 27/12

$$\mathcal{L}[f(t)] = \int_0^{\infty} f(t) e^{-st} dt, \quad s > 0$$

Ex) Find Laplace transform of function

$$f(t) = \begin{cases} e^{3t}, & 0 < t < 2 \\ 6, & t > 2 \end{cases}$$

by Defn $\mathcal{L}[f(t)] = \int_0^{\infty} f(t) \cdot e^{-st} dt$

$$= \int_0^2 e^{3t} \cdot e^{-st} dt + \int_2^{\infty} 6 e^{-st} dt$$

$$= \int_0^2 e^{(3-s)t} dt + 6 \lim_{b \rightarrow \infty} \int_2^b e^{-st} dt = \left[\frac{e^{(3-s)t}}{3-s} \right]_0^2 + 6 \lim_{b \rightarrow \infty} \left[\frac{e^{-st}}{-s} \right]_2^b$$

$$= \frac{1}{3-s} (e^{2(3-s)} - 1) + 6 \lim_{b \rightarrow \infty} \frac{1}{-s} (e^{-bt} - e^{-2t})$$

$$\Rightarrow F(s) = \mathcal{L}[f(t)] = \frac{1}{3-s} (e^{6-2s} - 1) + \frac{6}{s} e^{-2s}$$

* Ex) Evaluate the following

① $\mathcal{L}[\pi] = \frac{\pi}{s}$

② $\mathcal{L}[6e^{-2t} + 3t^4] = 6\mathcal{L}[e^{-2t}] + 3\mathcal{L}[t^4]$
 $= 6 \cdot \frac{1}{s+2} + \frac{3 \cdot 4!}{s^5}$

③ $\mathcal{L}[\sin(3t) - e^{3t}e^{-5t}]$
 $= \mathcal{L}[\sin(3t)] - \mathcal{L}[e^{3t}e^{-5t}]$
 $= \mathcal{L}[\sin(3t)] - e^{-5t} \mathcal{L}[e^{3t}]$
 $= \frac{3}{s^2+9} - e^{-5t} \cdot \frac{1}{s-3}$

④ $\mathcal{L}[\sin^2 t] \rightarrow \frac{1}{2} - \frac{1}{2} \cos(2t)$
 $= \mathcal{L}[\frac{1}{2}] - \frac{1}{2} \mathcal{L}[\cos(2t)]$
 $= \frac{1}{2s} - \frac{1}{2} \cdot \frac{s}{s^2+4}$

$$\textcircled{5} \mathcal{L}[\sin(2t) \cos(4t)] \quad \sin A \cos B = \frac{1}{2} \sin(A+B) + \frac{1}{2} \sin(A-B)$$

$$= \mathcal{L}\left[\frac{1}{2} \sin 3t\right] + \frac{1}{2} \mathcal{L}[\sin t]$$

$$= \frac{1}{2} \left(\frac{3}{s^2+9} + \frac{1}{s^2+1} \right)$$

$$\textcircled{6} \mathcal{L}[\sinh^2(5t)] \quad \left(\frac{e^{5t} - e^{-5t}}{2} \right)^2 = \frac{1}{4} (e^{10t} + 2 + e^{-10t})$$

$$= \frac{1}{4} (\mathcal{L}[2] + \mathcal{L}[e^{10t}] + \mathcal{L}[e^{-10t}])$$

$$= \frac{1}{4} \left(\frac{-2}{s} + \frac{1}{s+10} + \frac{1}{s-10} \right)$$

$$\textcircled{7} \mathcal{L}[\sin(3t+5)]$$

$$= \mathcal{L}[\sin(3t) \cos(5) + \cos(3t) \sin(5)]$$

$$= \cos(5) \mathcal{L}[\sin 3t] + \sin(5) \mathcal{L}[\cos 3t]$$

$$= \cos(5) * \frac{3}{s^2+9} + \sin(5) * \frac{5}{s^2+9}$$

* Using Laplace Transform to find Improper integral

Ex. Evaluate

$$\textcircled{1} \int_0^{\infty} e^{-st} \cdot t^3 dt = \mathcal{L}[t^3] = \frac{3!}{s^4}$$

$$\textcircled{2} \int_0^{\infty} e^{-3t} \sin(5t) dt = \mathcal{L}[\sin(5t)] \Big|_{s=3} = \frac{5}{s^2+25} \Big|_{s=3} = \frac{5}{34}$$

$$\textcircled{3} \int_0^{\infty} 2 \cos^2 t e^{-at} dt$$

$$= \mathcal{L}[2 \cos^2 t] \Big|_{s=a} = \mathcal{L}\left[2 \left(\frac{1}{2} + \frac{1}{2} \cos 2t\right)\right] \Big|_{s=a}$$

$$= \left[\mathcal{L}[1] + \mathcal{L}[\cos 2t] \right] \Big|_{s=a} = \left(\frac{1}{s} + \frac{s}{s^2+4} \right) \Big|_{s=a}$$

$$= \frac{1}{a} + \frac{a}{a^2+4}$$

(١) طريقة

$$\int_{-1}^0 e^t \cos t \, dt + \int_0^\infty e^{-t} \cos t \, dt$$

↓ ↓

الجزء دورى Laplace

الفرد $\Rightarrow U = t + 1 \Rightarrow t = U - 1$

↳ Laplace.

Rules of Laplace Transform:

Rule 1) Shifting rule.

$$\mathcal{L}[e^{at} f(t)] = \mathcal{L}[f(t)] \Big|_{s \rightarrow s-a} = F(s-a).$$

Ex) Evaluate

$$① \mathcal{L}[e^{at} t^3] = \mathcal{L}[t^3] \Big|_{s \rightarrow s-a} = \frac{6}{s^4} \Big|_{s \rightarrow s-a} = \frac{6}{(s-a)^4}$$

$$② \mathcal{L}[e^{-2t} \cosh(3t)] = \mathcal{L}[\cosh(3t)] \Big|_{s \rightarrow s+2} = \frac{s}{s^2+9} \Big|_{s \rightarrow s+2} = \frac{s+2}{(s+2)^2+9}$$

$$③ \mathcal{L}[\cosh(3t) \cosh(t)] = \frac{s+2}{s^2+4s-5}$$

$$\text{But } \cosh(3t) = \frac{e^{3t} + e^{-3t}}{2}$$

$$\rightarrow \frac{1}{2} \left[\mathcal{L}[\cosh(t)] \Big|_{s \rightarrow s-3} + \mathcal{L}[\cosh(t)] \Big|_{s \rightarrow s+3} \right]$$

$$= \frac{1}{2} \left[\frac{s}{s^2+1} \Big|_{s \rightarrow s-3} + \frac{s}{s^2+1} \Big|_{s \rightarrow s+3} \right] = \frac{s-3}{(s-3)^2+1} + \frac{s+3}{(s+3)^2+1}$$

Rule (2): Derivatives Rule

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} (F(s)) \rightarrow \text{where } \mathcal{L}[f(t)] = F(s)$$

$$\text{where } (-1)^n \rightarrow \begin{cases} +1, & n \text{ even} \\ -1, & n \text{ odd} \end{cases}$$

* Cases:

$$n=1 \rightarrow \mathcal{L}[t f(t)] = (-1)^1 \frac{d}{ds} \mathcal{L}[f(t)]$$

$$n=2 \rightarrow \mathcal{L}[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2} \mathcal{L}[f(t)]$$

$$\textcircled{1} \mathcal{L} \left[t \sin t \right] = -\frac{d}{ds} (\mathcal{L} [\sin t])$$

$$= -\frac{d}{ds} \left(\frac{1}{s^2+1} \right) = \frac{+2s}{(s^2+1)^2}$$

(2) Rule 3 and Rule 4 to be used

$$\textcircled{2} \mathcal{L} [t^2 e^t] = +\frac{d^2}{ds^2} (\mathcal{L} [e^t]) = \frac{d^2}{ds^2} \left(\frac{1}{s-1} \right)$$

$$= \frac{2}{(s-1)^3}$$

$(s-1)^{-1}$
 $\downarrow$
 $-(s-1)^{-2}$
 $\downarrow$
 $2(s-1)^{-3}$

$$* \mathcal{L} [t^n e^{at} f(t)] = (-1)^n \frac{d^n}{ds^n} [\mathcal{L} (f(t) e^{at})]$$

Deriv shift

$$= (-1)^n \frac{d^n}{ds^n} (F(s-a))$$

$$\textcircled{\text{Ex}} \mathcal{L} [t \cdot e^t \cdot \cos t] = -\frac{d}{ds} \mathcal{L} [\cos t] \downarrow$$

$s \rightarrow s+1$

$$= -\frac{d}{ds} \left(\frac{s}{s^2+1} \right) \downarrow$$

$s \rightarrow s+1$ $\frac{d}{ds} \left(\frac{s}{s^2+1} \right) = \frac{s^2-1}{(s^2+1)^2}$

Let's solve this problem

$$\textcircled{\text{Ex}} \text{ Find } \mathcal{L} [t^{\frac{3}{2}}] = \frac{\sqrt{\pi}}{2} s^{-\frac{3}{2}}$$

$$\textcircled{1} \mathcal{L} [t^{\frac{3}{2}}] = \mathcal{L} [t \cdot t^{\frac{1}{2}}]$$

$n=1$ shift

$$= -\frac{d}{ds} \mathcal{L} [t^{\frac{1}{2}}] = -\frac{d}{ds} \left(\frac{\sqrt{\pi}}{2} s^{-\frac{1}{2}} \right)$$

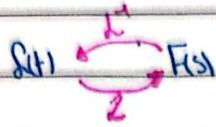
$$= \frac{3\sqrt{\pi}}{4} s^{-\frac{3}{2}}$$

$$\textcircled{2} \mathcal{L} [t^{\frac{3}{2}} e^{2t}] = \mathcal{L} [t^2 \cdot t^{\frac{1}{2}}] \downarrow$$

$s \rightarrow s-2$ $\frac{d^2}{ds^2} \left(\frac{\sqrt{\pi}}{2} s^{-\frac{1}{2}} \right) \downarrow$ $s \rightarrow s-2$

Inverse Laplace Transform.

* IF $F(s) = \mathcal{L}(f(t))$ then $f(t) = \mathcal{L}^{-1}[F(s)]$.



$$* \mathcal{L}^{-1}[F(s-a)] = e^{at} f(t).$$

* Ex) Evaluate:

$$\begin{aligned} \textcircled{1} \mathcal{L}^{-1}\left[\frac{2}{s^5} - \frac{3}{s}\right] &= \frac{1}{3 \cdot 4} \mathcal{L}^{-1}\left[\frac{4 \cdot 2 \cdot 1}{s^5}\right] - \mathcal{L}^{-1}\left[\frac{3}{s}\right] \\ &= \frac{1}{12} t^4 - 3 = f(t) \end{aligned}$$

$n+1=5$
 $n=4$

$$\textcircled{2} \mathcal{L}^{-1}\left[\frac{1}{s^2+64} - \frac{5}{s^3} + \frac{3}{s-2}\right]$$

$$= \frac{1}{8} \mathcal{L}^{-1}\left[\frac{8}{s^2+8^2}\right] - \frac{5}{2} \mathcal{L}^{-1}\left[\frac{2}{s^3}\right] + 3 \mathcal{L}^{-1}\left[\frac{1}{s-2}\right]$$

$$= \frac{1}{8} \sin(8t) - \frac{5}{2} t^2 + 3e^{2t}.$$

19-12-2023

(29) 2019

$$* \mathcal{L}[f(t)] = \int_0^{\infty} f(t) e^{-st} dt = F(s), \quad s > 0$$

Inverse Laplace Transform.

Ex) Evaluate

$$\textcircled{3} \mathcal{L}^{-1} \left[\frac{4}{(s-4)^5} \right] = e^{4t} f(t) = \frac{e^{4t}}{6} \cdot t^4$$

$$* f(t) = \frac{1}{6} \mathcal{L}^{-1} \left[\frac{4 \cdot 3!}{s^5} \right] = \frac{1}{6} \mathcal{L}^{-1} \left[\frac{4!}{s^5} \right] = \frac{1}{6} t^4$$

$$\textcircled{4} \mathcal{L}^{-1} \left[\frac{7s-1}{(s+1)(s+2)(s-3)} \right]$$

By Partial Fraction.

$$\frac{7s-1}{(s+1)(s+2)(s-3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-3}$$

$$\rightarrow 7s-1 = A(s+2)(s-3) + B(s+1)(s-3) + C(s+1)(s+2)$$

$$s=3 \rightarrow 20 = 20C \Rightarrow C=1$$

$$s=-2 \rightarrow -15 = 5B \Rightarrow B=-3$$

$$s=-1 \rightarrow -8 = -4A \Rightarrow A=2$$

$$= \mathcal{L}^{-1} \left[\frac{2}{s+1} \right] + \mathcal{L}^{-1} \left[\frac{-3}{s+2} \right] + \mathcal{L}^{-1} \left[\frac{1}{s-3} \right]$$

$$= e^{-t} \cdot 2 - 3e^{-2t} + e^{+3t}$$

$$\textcircled{5} \mathcal{L}^{-1} \left[\frac{5}{s^2 + 7s + 12} \right] \rightarrow \Delta = 49 - 4(1)(12) = 1$$

$$(s-4)(s-3)$$

$$\rightarrow 5 = A(s-3) + B(s-4)$$

$$s=3 \rightarrow 5 = -B \rightarrow B=-5$$

$$s=4 \rightarrow 5 = A$$

$$f(t) = 5e^{4t} - 5e^{3t}$$

$$\textcircled{6} \mathcal{L}^{-1} \left[\frac{S^2 + 9S + 2}{(S-1)^2(S+3)} \right] =$$

$$\frac{S^2 + 9S + 2}{(S-1)^2(S+3)} = \frac{A}{S-1} + \frac{B}{(S-1)^2} + \frac{C}{S+3}$$

$$\leadsto S^2 + 9S + 2 = A(S-1)(S+3) + B(S+3) + C(S-1)^2$$

$$\leadsto S=1 \leadsto 12 = 4B \Rightarrow \boxed{B=3}$$

$$\leadsto S=-3 \leadsto -16 = 16C \Rightarrow \boxed{C=-1}$$

$$\leadsto S=0 \leadsto 2 = -3A + 4(-1) \Rightarrow \boxed{A=2}$$

$$* \mathcal{L}^{-1} \left[\frac{2}{S-1} \right] + \mathcal{L}^{-1} \left[\frac{3}{(S-1)^2} \right] + \mathcal{L}^{-1} \left[\frac{-1}{S+3} \right]$$

$$\downarrow$$

$$2e^t$$

$$\downarrow$$

$$\boxed{e^t \cdot t}$$

$$\downarrow$$

$$-e^{-3t}$$

$$* \mathcal{L}^{-1} \left[\frac{3}{(S-1)^2} \right] = e^t \cdot f(t) = e^t \cdot t$$

$$f(t) = \frac{1}{S^2} = [t]$$

$$* \boxed{2e^t + e^t \cdot t - e^{-3t}}$$

$$\textcircled{7} \mathcal{L}^{-1} \left[\frac{1}{S(1+S^2)} \right] =$$

$$\frac{1}{S(1+S^2)} = \frac{A}{S} + \frac{BS+C}{1+S^2}$$

$$\leadsto 1 = A(1+S^2) + S(BS+C)$$

$$\leadsto S=0 \leadsto \boxed{1=A}, \quad B=-1, C=0$$

$$* \mathcal{L}^{-1} \left[\frac{1}{S} \right] + \mathcal{L}^{-1} \left[\frac{-S}{S^2+1} \right]$$

$$= 1 - \cos t$$

H.W)

$$① \mathcal{L}^{-1} \left[\frac{3}{(s^2+5)(s^2+9)} \right]$$

$$② \mathcal{L}^{-1} \left[\frac{7s^3 - 2s^2 + 6 - 3s}{s^2(s-2)s^3} \right]$$

$$③ \mathcal{L}^{-1} \left[\frac{s^2 + 9s + 2}{(s+1)^2(s+3)} \right]$$

± (b/2)² 2.54 11.11 *

8 *EX $\mathcal{L}^{-1} \left[\frac{3s+2}{s^2+2s+10} \right] =$

$$\mathcal{L}^{-1} \left[\frac{3s+2}{s^2+2s+1+9} \right] = \mathcal{L}^{-1} \left[\frac{3s+2}{(s+1)^2+9} \right]$$

$$\Rightarrow \mathcal{L}^{-1} \left[\frac{3s+1-1}{(s+1)^2+9} \right] + \mathcal{L}^{-1} \left[\frac{2}{(s+1)^2+9} \right]$$

$$\Rightarrow 3 \mathcal{L}^{-1} \left[\frac{(s+1)}{(s+1)^2+9} \right] + \mathcal{L}^{-1} \left[\frac{-3}{(s+1)^2+9} \right] + \frac{2}{3} \mathcal{L}^{-1} \left[\frac{3}{(s+1)^2+9} \right]$$

$$\Rightarrow 3e^{-t} \cos 3t + -e^{-t} \sin(3t) + \frac{2}{3} e^{-t} \sin(3t)$$

$$= 3e^{-t} \cos(3t) + \frac{1}{3} e^{-t} \sin(3t)$$

21-12-2023

(30) 5 p 12

$$\mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} f(t) dt, \quad s > 0$$

$f(t)$	$\mathcal{L}[f(t)] = F(s)$
0	$\frac{c}{s}$
e^{at}	$\frac{1}{s-a}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$\cosh(bt)$	$\frac{s}{s^2-b^2}$
t^n	$\frac{n!}{s^{n+1}}$

$$\mathcal{L}[e^{at} f(t)] = \mathcal{L}[f(t)] = F(s-a)$$

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} (\mathcal{L}[f(t)])$$

* Inverse Laplace Transform

Ex) Find $\mathcal{L}^{-1} \left[\ln \left(\frac{s^2+1}{s-1} \right) \right] =$

By derivative rule ($n=1$) $\rightarrow \mathcal{L}[t \cdot f(t)] = -\frac{d}{ds} \mathcal{L}[f(t)]$

① Let $\mathcal{L}^{-1} \left[\ln \left(\frac{s^2+1}{s-1} \right) \right] = f(t)$

② Take Laplace for both sides:

$$\ln(s^2+1) - \ln(s-1) = \mathcal{L}[f(t)]$$

③ Take $-\frac{d}{ds}(\dots)$ for both sides:

$$-\frac{d}{ds} (\ln(s^2+1) - \ln(s-1)) = -\frac{d}{ds} \mathcal{L}[f(t)]$$

$$-\left(\frac{2s}{s^2+1} - \frac{1}{s-1} \right) = \mathcal{L}[t \cdot f(t)]$$

$$-2 \frac{\cos(t)}{t} + \frac{e^t}{t} = \mathcal{L}[t \cdot f(t)] \quad \leftarrow \text{take } \mathcal{L}^{-1}$$

$$-2 \cos t + e^t = t \cdot f(t) \quad \times t \Rightarrow f(t) = \frac{-2 \cos t + e^t}{t}$$

Ex) $\mathcal{L}^{-1}[\tan^{-1}(3s)] =$

$$\mathcal{L}^{-1}[\tan^{-1}(3s)] = \mathcal{L}(f) \Rightarrow \mathcal{L}[\tan^{-1}(3s)] = \mathcal{L}(f)$$

$$\frac{-d}{ds} [\tan^{-1}(3s)] = \frac{d}{ds} \mathcal{L}(f)$$

$$\frac{-3}{3^2(1/3 + s^2)} = \mathcal{L}(f)$$

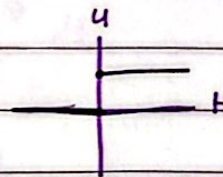
$$-\mathcal{L}[\sin(\frac{1}{3}t)] = \mathcal{L}(f) \Rightarrow \text{take } \mathcal{L}^{-1}$$

$$\frac{-\sin(\frac{1}{3}t)}{t} = \frac{1}{t} \cdot \mathcal{L}(f) \Rightarrow f(t) = \frac{-\sin(\frac{1}{3}t)}{t}$$

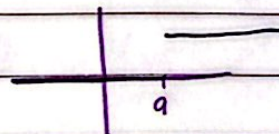
* Unit Step Function.

Defn:

$$① u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$



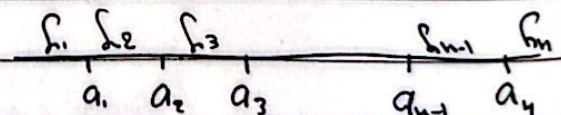
$$② u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$$



Note:

If $a < a_1 < a_2 < a_3 < \dots < a_{n-1} < a_n$

$$\text{let } f(t) = \begin{cases} f_1(t), & t < a_1 \\ f_2(t), & a_1 \leq t < a_2 \\ f_3(t), & a_2 \leq t < a_3 \\ \vdots & \vdots \\ f_{n-1}(t), & a_{n-1} \leq t < a_n \\ f_n(t), & a_n \leq t \end{cases}$$



* Then...

* Then, $f(t)$ can be written as

$$f(t) = f_1(t) + (f_2 - f_1) \cdot u(t-a_1) + (f_3 - f_2) \cdot u(t-a_2) + \dots + (f_n - f_{n-1}) \cdot u(t-a_n)$$

* Ex

$$f(t) = \begin{cases} e^t, & t < 1 \\ t, & 1 < t < 2 \\ 1, & 2 < t \end{cases} \Rightarrow f(t) = e^t + (t - e^t) \cdot u(t-1) + (1-t) \cdot u(t-2)$$

* $\mathcal{L}[u(t-a)]$

* $\mathcal{L}[u(t-a)g(t)]$ $\rightarrow$ $f(s) = \frac{e^{-as}}{s}$

$\Rightarrow$ Laplace for Unit Step Function:

$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$$

By Defn: $\mathcal{L}[u(t-a)] = \int_0^\infty u(t-a) \cdot e^{-st} dt$

(a) $\Rightarrow$ $\mathcal{L}[u(t-a)] = \int_0^a 0 \cdot e^{-st} dt + \int_a^\infty 1 \cdot e^{-st} dt$

$$= \int_a^\infty e^{-st} dt = \lim_{K \rightarrow \infty} \int_a^K e^{-st} dt = \lim_{K \rightarrow \infty} \left[\frac{e^{-st}}{-s} \right]_a^K$$

$$= \lim_{K \rightarrow \infty} \frac{1}{-s} (e^{-sK} - e^{-as}) = \frac{e^{-as}}{s}$$

* $\mathcal{L}[u(t-a)] = \frac{e^{-as}}{s}$

* $\mathcal{L}[u(t-a)f(t)] = e^{-as} \mathcal{L}[f(t+a)]$

Unit Step Function

$$u(t-a) = \begin{cases} 0 & , t < a \\ 1 & , t \geq a \end{cases}$$

$$u(t) \equiv H(t) \equiv 1$$

$$\textcircled{1} \mathcal{L}[u(t-a)] = \frac{e^{-as}}{s}$$

$$\textcircled{2} \mathcal{L}[u(t-a)f(t)] = e^{-as} \mathcal{L}[f(t+a)]$$

$$* f(t) \text{ cont } [0, \infty] \rightarrow \mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} f(t) dt$$

Ex) Find $\mathcal{L}[f(t)]$ where

$$f(t) = \begin{cases} 2 & , 0 < t \leq 1 \\ 0 & , 1 < t \leq 2 \\ e^t & , t > 2 \end{cases}$$

By Notes

$$f(t) = 2 + (0-2)u(t-1) + (e^t-0)u(t-2)$$

$$\mathcal{L}[f(t)] = \mathcal{L}[2 - 2u(t-1) + e^t u(t-2)]$$

$$\rightarrow \mathcal{L}[f(t)] = \mathcal{L}[2] - 2\mathcal{L}[u(t-1)] + \mathcal{L}[e^t u(t-2)]$$

$$= \frac{2}{s} - 2 \cdot \frac{e^{-s}}{s} + e^{-2s} \mathcal{L}[e^{t+2}]$$

$$\hookrightarrow e^{t+2} = e^2 \cdot e^t$$

$$= \frac{2}{s} - \frac{2e^{-s}}{s} + e^{-2s} \cdot e^2 \mathcal{L}[e^t]$$

$$= \frac{2}{s} - \frac{2e^{-s}}{s} + e^{2-2s} \cdot \frac{1}{s-1}$$

(H.w) Find $\mathcal{L}[f(t)]$ where $f(t) = \begin{cases} 1 & , 0 \leq t \leq 1 \\ -1 & , 1 \leq t \leq 2 \\ 2 & , t \geq 2 \end{cases}$

* Laplace Inverse and Unit Step

* III $\mathcal{L}^{-1} \left[\frac{e^{-as}}{s} \right] = u(t-a)$

* [2] $\mathcal{L}^{-1} [e^{-as} \cdot f(s)] = f(t-a) u(t-a)$

* Ex) $\mathcal{L}^{-1} \left[\frac{e^{-\pi s}}{s} \right] = u(t-\pi)$

② $\mathcal{L}^{-1} \left[\frac{e^{-2s}}{s-4} \right] \rightsquigarrow a=2, F(s) = \frac{1}{s-4}$

$\mathcal{L}(t) = \mathcal{L}^{-1} \left[\frac{1}{s-4} \right] = e^{4t}$

$\Rightarrow f(t-2) u(t-2) = e^{4(t-2)} u(t-2)$

* $f(t-2) = \begin{cases} 0, & t < 2 \\ e^{4(t-2)}, & t > 2 \end{cases}$

$\hookrightarrow u(t-2) = \begin{cases} 0, & t < 2 \\ 1, & t \geq 2 \end{cases}$

③ $\mathcal{L}^{-1} \left[\frac{e^{-\frac{\pi}{2}s} \cdot s}{s^2+9} \right]$

$a = \frac{\pi}{2}, F(s) = \frac{s}{s^2+9}$

* $\mathcal{L}(t) = \mathcal{L}^{-1} F(s) = \mathcal{L}^{-1} \left[\frac{s}{s^2+9} \right] = \cos 3t$

$\Rightarrow f(t-\frac{\pi}{2}) u(t-\frac{\pi}{2}) = \cos 3(t-\frac{\pi}{2}) u(t-\frac{\pi}{2})$

$\hookrightarrow \cos(A-B) = \cos A \cos B + \sin A \sin B$

$\rightsquigarrow (\cos 3t * \cos \frac{3\pi}{2} + \sin 3t * \sin \frac{3\pi}{2}) u(t-\frac{\pi}{2})$

$\rightsquigarrow -\sin 3t \cdot u(t-\frac{\pi}{2})$

④ $\mathcal{L}^{-1} \left[e^{-s} \frac{3s^2+s}{s^3} \right] \Rightarrow a=1, F(s) = \frac{3}{s} + \frac{1}{s^2}$

$\hookrightarrow f(t-1) u(t-1) = 3 + (t-1) u(t-1)$
 $(2+t) u(t-1)$

$\hookrightarrow \mathcal{L}(t) = \mathcal{L}^{-1} F(s)$
 $= 3 + t$

$$Q_1) \mathcal{L} [t \cos(2t-2) u(t-1)] \quad \rightarrow a=1$$

$$\mathcal{L} [f(t) u(t-a)] = e^{-as} \mathcal{L} [f(t+a)]$$

$$= e^{-s} \mathcal{L} [f(t+1)] = e^{-s} \left[\frac{-(4+s^2)}{(s^2+4)^2} + \frac{s}{s^2+4} \right]$$

$$* f(t) = t \cos(2t-2)$$

$$* f(t+1) = (t+1) \cos(2t)$$

$$* \mathcal{L} [t \cos(2t)] + \mathcal{L} [\cos(2t)] = \mathcal{L} [f(t+1)]$$

↓

* by derivative Rule

$$\mathcal{L} [t \cos(2t)] = -\frac{d}{ds} \mathcal{L} [\cos(2t)]$$

$$= -\frac{d}{ds} \left[\frac{s}{s^2+4} \right] = -\frac{(s^2+4) - s \cdot 2s}{(s^2+4)^2}$$

$$= \frac{s^2-4}{(s^2+4)^2}$$

$$\rightarrow \mathcal{L} [f(t+1)] = \frac{s^2-4}{(s^2+4)^2} + \frac{s}{s^2+4}$$

$$* \mathcal{L} [5^t (t^2-9) u(t-3)] \quad (a=3)$$

$$e^{t \ln 5} * \mathcal{L} [e^{\ln 5 \cdot t} (t^2-9) u(t-3)]$$

$$= \mathcal{L} [(t^2-9) u(t-3)]$$

$$s \rightarrow s - \ln 5$$

$$= e^{-3s} \mathcal{L} [f(t+3)]$$

$$e^{-3s} \mathcal{L} [t^2+6t]$$

If $f(t)$ cont or Piecewise cont $\rightarrow$ have finite dis cont. pt.

* $f(t) = e^t, t^n, \sin(at) \dots$

* $f(t) = \begin{cases} f_1 \\ f_2 \\ \vdots \\ f_n \end{cases} \rightarrow$ Use Unit Step fn To find Laplace

* $\mathcal{L}[f(t)] = F(s)$

* $\mathcal{L}^{-1}[F(s)] = f(t)$

$\rightarrow$ IVP $\equiv$ DE + IC

$\rightarrow y(x_0) = y_0 / y'(x_0) = y_1 / y''(x_0) = y_2 / \dots / y^{(n-1)}(x_0) = y_{n-1}$

* Solve IVP by Using Laplace Transform.

By Using Laplace Transform:

① General Sol without any const. (C_1, C_2)

② No need y_p for non Homog. Part

$\rightarrow$ Let $y(t)$ is the general Sol to D.E

Where $\mathcal{L}[y(t)] = Y(s) \Leftrightarrow \mathcal{L}^{-1}[Y(s)] = y(t)$.

* Laplace Transform derivative:

$\mathcal{L}[y^{(n)}] = s^n Y(s) - s^{n-1} y(x_0) - s^{n-2} y'(x_0) - \dots - y^{(n-1)}(x_0)$

* $n \rightarrow 1 \rightarrow \mathcal{L}[y'] = s Y(s) - y(x_0)$

* $n \rightarrow 2 \rightarrow \mathcal{L}[y''] = s^2 Y(s) - s y(x_0) - y'(x_0)$

Cauchy Second order * $at^2y'' + bt'y' + cy = f(t)$.

* $\mathcal{L}[t^2 y''] = (-1)^2 \frac{d^2}{ds^2} \mathcal{L}[y'']$ Derivative Rule
 $= \frac{d^2}{ds^2} (S^2 Y - S y(x_0) - y'(x_0))$ $\mathcal{L}[f(t)] = (-1)^n \frac{d^n}{ds^n} \mathcal{L}[f(t)]$
 $= \frac{d}{ds} (S^2 Y' + 2SY - y(x_0)) \Rightarrow S^2 Y'' + Y'(2S) + 2Y + S(2Y')$
 $= S^2 Y'' + 4SY' + 2Y.$

* $\mathcal{L}[ty'] = -\frac{d}{ds} \mathcal{L}[y'] = -\frac{d}{ds} (SY - y(x_0))$
 $= (SY' + Y) * (-1) = -SY' - Y$

Steps

[1] Take Laplace Transform for both side of D.E and use linearity $\{ \mathcal{L}[y']$, $\mathcal{L}[y'']$ and IC

[2] Solve for $Y(s) =$ (in terms of s and $y(x_0)$)

[3] Take Laplace Inverse to find General Sol $\mathcal{L}^{-1}[Y(s)] = y(t).$

Ex) Solve IVP 8

$$y'' - y' - 6y = 0, \quad y(0) = 2, \quad y'(0) = -1$$

$$\text{char. eqn} = r^2 - r - 6 = 0 \Rightarrow (r-3)(r+2) = 0$$

$$r = 3 \quad / \quad r = -2$$

$$y(x) = C_1 e^{3x} + C_2 e^{-2x}$$

$$y'(x) = 3C_1 e^{3x} - 2C_2 e^{-2x}$$

$$\Rightarrow y(0) = 2 \Rightarrow (2 = C_1 + C_2) \quad * 3$$

$$\Rightarrow y'(0) = -1 \Rightarrow -1 = 3C_1 - 2C_2$$

$$C_1 = \frac{3}{5}, \quad C_2 = \frac{7}{5}$$

$$y(x) = \frac{3}{5} e^{3x} + \frac{7}{5} e^{-2x}$$

By Laplace Transform

$$\text{II} \quad \mathcal{L}[y''] - \mathcal{L}[y'] - 6\mathcal{L}[y] = \mathcal{L}[0]$$

$$s^2 Y - s y(0) - y'(0) - (sY - y(0)) - 6Y = 0$$

$$s^2 Y - 2s + 1 - sY + 2 - 6Y = 0$$

$$Y(s^2 - s - 6) - 2s + 3 = 0$$

$$\mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}\left[\frac{2s-3}{s^2-s-6}\right] \Rightarrow y(t) = \mathcal{L}^{-1}\left[\frac{2s-3}{s^2-s-6}\right]$$

$$\frac{2s-3}{s^2-s-6} = \frac{A}{(s-3)} + \frac{B}{s+2}$$

$$2s-3 = (s+2)A + (s-3)B \Rightarrow B = \frac{7}{5}, \quad A = \frac{3}{5}$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{\frac{3}{5}}{s-3}\right] + \mathcal{L}^{-1}\left[\frac{\frac{7}{5}}{s+2}\right]$$

$$= \frac{3}{5} e^{3t} + \frac{7}{5} e^{-2t}$$

Ex 2) Solve : $y'' + y = t$, $y(0) = 0$, $y'(0) = 1$

* By Laplace transform :

[1] $\mathcal{L}[y''] + \mathcal{L}[y] = \mathcal{L}[t]$

$$s^2 Y - s y(0) - y'(0) + Y = \frac{1}{s^2}$$

$$s^2 Y - 1 + Y = \frac{1}{s^2}$$

[2] $Y(s^2 + 1) = 1 + \frac{1}{s^2} \rightarrow \frac{s^2 + 1}{s^2} = Y(s^2 + 1)$

$$Y = \frac{1}{s^2}$$

[3] $\mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right]$

$y(t) = t$

جوابه حليها زي (4)

$$\boxed{1} \quad \mathcal{L}[u(t-a)] = \frac{e^{-as}}{s}$$

$$\boxed{2} \quad \mathcal{L}[f(t) u(t-a)] = e^{-as} \mathcal{L}[f(t+a)] \quad \text{Laplace transform of } f(t) \text{ shifted by } a$$

$$* \quad \mathcal{L}^{-1}\left[\frac{e^{-as}}{s}\right] = u(t-a)$$

$$* \quad \mathcal{L}^{-1}[e^{-as} F(s)] = u(t-a) f(t-a) \quad \text{(Laplace transform of } f(t) \text{ shifted by } a)$$

Using Laplace Transform to Solve IVP:

$$* \quad \mathcal{L}[y'] = sY - y(x_0)$$

$$* \quad \mathcal{L}[y''] = s^2 Y - sy(x_0) - y'(x_0)$$

→ take Laplace to both sides. ①

→ solve for $Y(s) = \dots$ ②

→ $y(t) = \mathcal{L}^{-1}[Y(s)]$ ③

Ex

$$y' - 2y = g(t), \quad y(0) = 0$$

$$\text{where } g(t) = \begin{cases} 0, & t < 1 \\ 1, & t \geq 1 \end{cases}$$

$$* \quad y' - 2y = u(t-1)$$

$$\mathcal{L}[y'] - 2\mathcal{L}[y] = \mathcal{L}[u(t-1)]$$

$$sY - y(0) - 2Y = \frac{e^{-s}}{s}$$

$$Y(s-2) = \frac{e^{-s}}{s}$$

$$Y = \frac{e^{-s}}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}$$

$$A = -\frac{1}{2}, \quad B = \frac{1}{2}$$

$$\mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}\left[\frac{-1/2}{s}\right] + \mathcal{L}^{-1}\left[\frac{1/2}{s-2}\right]$$

$$g(t) = -\frac{1}{2} + \frac{1}{2} e^{2t}$$

$$y(t) = u(t-1) g(t-1) = u(t-1) \left(-\frac{1}{2} + \frac{1}{2} e^{2(t-1)}\right)$$

Ex

$$y'' + 3y' + 2y = g(t), \quad y(0) = 2, \quad y'(0) = 1$$

$$g(t) = \begin{cases} e^{-t}, & t < 3 \\ 1, & t \geq 3 \end{cases}$$

$$* y'' + 3y' + 2y = e^{-t} + (1 - e^{-t}) u(t-3)$$

$$\hookrightarrow \mathcal{L}[y''] + 3\mathcal{L}[y'] + 2\mathcal{L}[y] = \mathcal{L}[e^{-t}] + \mathcal{L}[u(t-3)] + \mathcal{L}[e^{-t} u(t-3)]$$

$$\hookrightarrow s^2 Y - s y(0) - y'(0) + 3sY - 3y(0) + 2Y = \frac{1}{s+1} + \frac{e^{-3s}}{s} + e^{-3s} \mathcal{L}[e^{-t} u(t-3)]$$

$$\hookrightarrow Y(s^2 + 3s + 2) - 2s - 7 = \frac{1}{s+1} + \frac{e^{-3s}}{s} + e^{-3s} \cdot e^{-3} \mathcal{L}[e^{-t}]$$

$$Y(s^2 + 3s + 2) - 2s - 7 = \frac{1}{s+1} + \frac{e^{-3s}}{s} + \frac{e^{-3s-3}}{s+1}$$

$$Y = \frac{\frac{1}{s+1} + \frac{e^{-3s}}{s} + \frac{e^{-3s-3}}{s+1} + 2s + 7}{s^2 + 3s + 2}$$

$$\hookrightarrow (s+2)(s+1)$$

$$Y = \frac{1}{(s+1)^2(s+2)} + \frac{e^{-3s}}{s(s+1)(s+2)} + \frac{e^{-3s-3}}{(s+1)^2(s+2)} + \frac{2s+7}{(s+2)(s+1)}$$

نفسه كسور $\mathcal{L}^{-1}$

Solve IVP by using Laplace transform.

$$\mathcal{L}[y'] = sY - y(0)$$

$$\mathcal{L}[y''] = s^2 Y - sy(0) - y'(0)$$

Note: To use Method of Laplace Transform the IC should be at zero ($x_0=0$)

Ex] $y(x)=2 \Rightarrow x=t-\pi$

$$* \mathcal{L}[u(t-a)] = \frac{e^{-as}}{s} \Rightarrow \mathcal{L}^{-1}\left[\frac{e^{-as}}{s}\right] = u(t-a)$$

$$* \mathcal{L}[f(t) u(t-a)] = e^{-as} \mathcal{L}[f(t+a)] \Rightarrow \mathcal{L}^{-1}[e^{-as} F(s)] = u(t-a) f(t-a)$$

Ex) Use Laplace Transform to Solve IVP

$$y'' + 4y = g(t), \quad y(0) = -1, \quad y'(0) = 0$$

$$g(t) = \begin{cases} t, & t \leq 2 \\ 5, & t > 2 \end{cases} \Rightarrow g(t) = t + (5-t)u(t-2)$$

$$* \mathcal{L}[y''] + 4\mathcal{L}[y] = \mathcal{L}[t] + 5\mathcal{L}[u(t-2)] - \mathcal{L}[tu(t-2)]$$

$$\hookrightarrow s^2 Y - s(-1) - 0 + 4Y = \frac{1}{s^2} + \frac{5e^{-2s}}{s} - e^{-2s}(\mathcal{L}[t] + \mathcal{L}[2])$$

$$\hookrightarrow \frac{1}{s^2} + \frac{5e^{-2s}}{s} - e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right) = s$$

$$\mathcal{L}[Y] = \frac{\frac{1}{s^2} + \frac{5e^{-2s}}{s} - e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right) - s}{s^2 + 4}$$

Find the inverse Laplace transform

If $Y(s) = \mathcal{L}[y(t)]$ where $y(t)$ is the general
Sol. of $y'' + y = 1$, Find $Y(s)$.

* Let $y(0) = a$, $y'(0) = b$

$$\rightarrow \mathcal{L}[y''] + \mathcal{L}[y] = \mathcal{L}[1]$$

$$\rightarrow s^2 Y - s y(0) - y'(0) + Y = \frac{1}{s}$$

$$\rightarrow Y(s^2 + 1) = \frac{1}{s} + as + b$$

$$\rightarrow Y = \left(\frac{1}{s(s^2 + 1)} + \frac{as + b}{s^2 + 1} \right)$$

H.w

Solve:-

① $y'' + y = g(t)$, $y(0) = 0$, $y'(0) = 1$

$$g(t) = \begin{cases} 0, & t < 3 \\ -2, & 3 \leq t < 6 \\ 0, & t \geq 6 \end{cases}$$

② $y'' + t y' - 2y = -2$, $y(0) = 0$, $y'(0) = 0$

③ $y'' - 6y' + 9y = t^2 e^{3t}$, $y(0) = 2$, $y'(0) = 7$

④ $y'' - 6y' + 5y = 3e^{2t}$, $y(0) = 2$, $y'(0) = 3$

Ex) If $f(0) = 6$ and $F(s) = \mathcal{L}[f(t)]$
Find $\mathcal{L}[(t-1)e^{3t-3}u(t-1)f''(t-1)]$

$$\Rightarrow \mathcal{L}[f(t)u(t-a)] = e^{-as} * \mathcal{L}[f(t+a)]$$

$$\mathcal{L}[(t-1)e^{3(t-1)}f''(t-1)u(t-1)]$$

$$f(t+1) = t+1-1 e^{3(t+1-1)} f''(t+1-1) = t e^{3t} f''(t)$$

$$\Rightarrow \mathcal{L}[t e^{3t} f''(t)]$$

by shifting $\mathcal{L}[t f''(t)] \downarrow s \rightarrow s-3$

by derivative $\left(-\frac{d}{ds} \mathcal{L}[f''(t)] \right) \downarrow s \rightarrow s-3$

$$\frac{-d}{ds} (s^2 F - s f(0) - f'(0)) \downarrow s \rightarrow s-3$$

$$\Rightarrow -(s^2 F' + F(2s) - 6 - 2 \cdot 0) \downarrow s \rightarrow s-3$$

$$\Rightarrow -(s-3)^2 F'(s-3) - 2(s-3)F(s-3) + 6 = \mathcal{L}[f(t+1)]$$

$$= e^{-as} * (- (s-3)^2 F'(s-3) - 2(s-3)F(s-3) + 6)$$

Ex) If $s > 0$, Find $\int_0^{\infty} e^{st - \ln 3} t^3 dt$

$$\Rightarrow e^{-\ln 3} \int_0^{\infty} e^{st} t^3 dt = 3^{-1} \mathcal{L}[t^3] = \frac{1}{3} * \frac{3!}{s^4} = \frac{2}{s^4}$$

2-1-2024

(35) Example

$$\mathcal{L}[f(t)] = \int_0^{\infty} f(t) e^{-st} dt$$

$f(t)$	$F(s) = \mathcal{L}[f(t)]$	
C	C/s	* $\mathcal{L}[u(t-a)] = \frac{e^{-as}}{s}$
e^{at}	$1/s-a$	* $\mathcal{L}[u(t-a) f(t)] = e^{-as} \mathcal{L}[f(t-a)]$
$\sin at$	a/s^2+a^2	* $\mathcal{L}[e^{-as}/s] = u(t-a)$
$\cos bt$	s/s^2+b^2	* $\mathcal{L}[e^{-as} F(s)] = u(t-a) f(t-a)$
t^n	$n! / s^{n+1}$	* $\mathcal{L}[y'] = sY - y(x_0)$
$e^{at} f(t)$	$F(s-a)$	* $\mathcal{L}[y''] = s^2Y - sy(x_0) - y'(x_0)$
$t^n f(t)$	$(-1)^n \frac{d^n}{ds^n} \mathcal{L}[f(t)]$	

$$\Rightarrow \mathcal{L}\left[\frac{t^3}{3e^{-6t}}\right] = \frac{1}{3} \mathcal{L}[t^3 e^{6t}] = \frac{1}{3} \mathcal{L}[t^3] \Big|_{s \rightarrow s-6}$$

$$= \frac{1}{3} \left(\frac{3!}{s^4} \right) \Big|_{s \rightarrow s-6} = \frac{2}{(s-6)^4}$$

$$\Rightarrow \mathcal{L}[e^{-\pi t} u(t-2)] = e^{-2s} \mathcal{L}[f(t+2)]$$

$\downarrow f(t) \quad ; \quad f(t+2) = e^{-\pi(t+2)} = e^{-\pi t} \cdot e^{-2\pi}$

$$= e^{-2s} \cdot e^{-2\pi} \mathcal{L}[e^{-\pi t}] = e^{-2(\pi+s)} \cdot \frac{1}{s+\pi}$$

$$\Rightarrow \text{If } \mathcal{L}[y(t)] = Y(s) \text{ Find } \mathcal{L}[t^2 y'' - 2y]$$

$$= \mathcal{L}[t^2 y''] - 2\mathcal{L}[y] = \frac{d^2}{ds^2} [s^2 Y - s y(0) - y'(0)] - 2Y$$

$$\text{If } s^2 Y + 2sY - y(0) \Rightarrow s^2 Y + 2sY + 2sY + 2Y$$

$$= s^2 Y + 4sY + 2Y - 2Y = s^2 Y + 4sY$$

$$\mathcal{L}^{-1} \left[\frac{e^{-2\pi s}}{s^2 - 4s + 5} \right] = u(t - 2\pi) f(t - 2\pi)$$

$$F(s) = \frac{1}{s^2 - 4s + 5} = \frac{1}{(s-2)^2 + 1} \Rightarrow e^{2t} \sin t$$

$$\Rightarrow u(t - 2\pi) e^{2(t-2\pi)} \sin(t - 2\pi)$$

$$\rightarrow \begin{cases} 0 & , t < 2\pi \\ e^{2(t-2\pi)} \sin(t - 2\pi) & , t > 2\pi \end{cases}$$

$$\mathcal{L}^{-1} \left[4 \tan^{-1} \left(\frac{y}{s} \right) \right] = f(t)$$

$$\mathcal{L}[t f(t)] = -\frac{d}{ds} \mathcal{L}[f(t)]$$

$$\mathcal{L}^{-1} \left[4 \tan^{-1} \left(\frac{y}{s} \right) \right] = \mathcal{L}[f(t)]$$

$$\left(4 \tan^{-1} \frac{y}{s} = \mathcal{L}[f(t)] \right) * -\frac{d}{ds}$$

$$= 4 * \frac{-\frac{y}{s^2}}{1 + \left(\frac{y}{s}\right)^2} = \mathcal{L}[t f(t)]$$

$$\rightarrow \frac{16}{s^2 \left(1 + \frac{16}{s^2} \right)} = \frac{16}{s^2 + 16}$$

$$\mathcal{L}^{-1} * \left(4 * \frac{y}{s^2 + 16} = \mathcal{L}[t f(t)] \right) \Rightarrow t.f(t) = 4 \mathcal{L}^{-1} \left[\frac{y}{s^2 + 16} \right]$$

$$t.f(t) = 4 \sin 4t \Rightarrow f(t) = \frac{4}{t} \sin 4t$$

6 Find $f(t)$ for : $S^2 F(s) - 4(F(s)) = \frac{5}{s+1}$

$$F(s) (S^2 - 4) = \frac{5}{S^2 + 1}$$

$$\mathcal{L}^{-1} \left(F(s) = \frac{5}{(S^2 + 1)(S^2 - 4)} \right) \Rightarrow f(t) = \mathcal{L}^{-1} \left[\frac{5}{(S^2 + 1)(S^2 - 4)} \right]$$

$$\frac{5}{(S+1)(S-2)(S+3)} = \frac{A}{S+1} + \frac{B}{S-2} + \frac{C}{S+3}$$

$$5 = A(S-2)(S+3) + B(S+1)(S+3) + C(S-2)(S+1)$$

$$S = -1 \Rightarrow 5 = A(-3)(2) \Rightarrow A = -\frac{5}{6}$$

$$S = 2 \Rightarrow 5 = B(3)(5) \Rightarrow B = \frac{1}{3}$$

$$S = -3 \Rightarrow 5 = C(-5)(-2) \Rightarrow C = \frac{1}{2}$$

$$* \quad \cancel{e^{5t}} \quad -\frac{5}{6} e^{+} + \frac{1}{3} e^{2t} + \frac{1}{2} e^{-3t}$$

$$\nabla \quad \int_0^{\infty} e^{-4t} \sinh(3t) dt = \mathcal{L}[\sinh(3t)] \Big|_{s=4}$$

$$= \frac{3}{s^2 - 9} \Big|_{s=4} = \frac{3}{16-9} = \frac{3}{7}$$

8 If $\mathcal{L} \left[\frac{1}{\sqrt{t}} \right] = \sqrt{\frac{2}{s}}$

$$\text{Find } \mathcal{L} \left[\frac{e^{-\pi t}}{\sqrt{8t}} \right] = \frac{1}{\sqrt{8}} \mathcal{L} \left[e^{-\pi t} \cdot \frac{1}{\sqrt{t}} \right]$$

$$= \frac{1}{2\sqrt{2}} \mathcal{L} \left[\frac{1}{\sqrt{t}} \right] \Big|_{s \rightarrow s+\pi} = \frac{1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{s}} \Big|_{s \rightarrow s+\pi}$$

$$= \frac{1}{2} \frac{1}{\sqrt{s+\pi}}$$