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دفتر :

معادلات تفاضلية عادية

Ordinary Differential Equations

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(2)

[2] partial differential equation (PDE).

معتمد على اثنين او اكثر متغيرات مستقلة $\rightarrow$ متغيرين، اكثر

• one or more dep. v with respect to two or more ind. v.

Ex 3. 1] $\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = e^x$

2] $\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial z^2} = \frac{\partial^2 y}{\partial \theta^2}$

② Order $\rightarrow$ classification by order.

• The order of a DE is the order of the highest derivative in the equation.

Ex 3. 1] $\frac{d^2 y}{dx^2} + 5y' - 4y = \sin x$ (second order).

2] $y'' + 5(y')^3 - 4y = e^x$ (second order).

3] $5t + y^{(4)} + 3t^2 y'' - \sin t y^6 = 0$ (fourth order)

4] $y^{(10)} [y^{(1)} - y^{(2)} + y^3 + 1] = 0$ (eleventh order).

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③ linearity → classification as linear or nonlinear.

• An equation is linear if it can be written in the form-

$$a_n x(y)^n + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = g(x)$$

شروط الخطية - يجب أن تكون المتغير ومتكافئة الأس (المتوى)
المعادلة ومتكافئة بمتغيرات، ومتكافئة موجودة ذلك (الم)
أو (log) أو (sin) أو (cos) أو ...، ومتكافئة على
متغير ثاني أو ثابت.

Ex:- 1] $y'' + 2xy' - 3x^2y = e^x$ linear in y .

2] $3r\theta' + 2\theta = r$ linear in θ .

3] $3y x'' + 2x''' = 2x$ linear in x .

4] $2xy'' + y' \cdot y^{(3)} + 2xy = 0$ non linear.

5] $2xy' + 2\sin(xy) = e^x$ non linear.

6] $\frac{dx}{dy} = \frac{3}{x - e^y}$ is not linear in y , but linear in x because:-

$$\frac{dx}{dy} = \frac{x - e^y}{3} \rightarrow \frac{dx}{dy} = \frac{x}{3} - \frac{e^y}{3}$$

$$\frac{dx}{dy} - \frac{1}{3}x = -\frac{1}{3}e^y$$

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Solution of DE

Explicit solution

Implicit solution

Ex:- $y = f(x)$

... y is a function of x

Ex:- $x^2 + y^2 = 4$

... y is not a function of x

Ex:- show that $y = e^{2x}$ is an explicit solution to $y' - 2y = 0$?

Sol:- $y' - 2y = 0 \rightarrow y' = 2e^{2x}, y = e^{2x}$
 $\downarrow$
sub: $y = e^{2x} \rightarrow 2e^{2x} - 2e^{2x} = 0$ (✓)

* Solution $\rightarrow$ y is a function of x

Ex:- show that $x^2 - y^2 = 4$ is an implicit solution to $y \cdot \frac{dy}{dx} - 4x = 0$?

Sol:- $x^2 - y^2 = 4 \rightarrow 2x - 2y \cdot y' = 0$
 $\downarrow$
div: $\rightarrow 8x - 2y \cdot y' = 0$ / -2
 $-4x + y \cdot \frac{dy}{dx} = 0 \rightarrow$ ~~$4x - y \cdot \frac{dy}{dx} = 0$~~
 $\downarrow$
 $y \cdot \frac{dy}{dx} - 4x = 0$ (✓)

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* Remark 3: The ODE of n^{th} order has the form: $y^{(n)} = f(x, y, y', y'', \dots, y^{(n-1)})$

• Initial value problem (I.V.P)

"Solve the following I.V.P: $y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$

where $y(x_0) = y_0, y'(x_0) = y_1, y''(x_0) = y_2, \dots, y^{(n-1)}(x_0) = y_{n-1}$ "

Ex 3: Suppose that $y(x)$ is a solution of the I.V.P
 $y' = x^2 + y^2$ with $y(1) = 2$, find $y'(1), y''(1), y'''(1)$:

Sol:

$$\begin{aligned} y' &= x^2 + y^2 \rightarrow y'(1) = 1^2 + y^2(1) = 1 + 2^2 = \underline{5} \\ y'' &= 2x + 2y \cdot y' \rightarrow y''(1) = 2(1) + 2(2) \cdot 5 = \underline{22} \\ y''' &= 2 + (2y)(y'') + (y')(2 \cdot y') = y'''(1) = 2 + 2(2)(22) + 5(2 \cdot 5) \\ &= \underline{140} \end{aligned}$$

* Interval of validity.

* Thm 3: Consider the following I.V.P
 $y'(x) + p(x)y(x) = q(x), y(x_0) = y_0,$

if $p(x)$ and $q(x)$ are continuous function on a open interval $\alpha < x < \beta$ and the interval contains x_0 , then there is a Unique solution of the I.V.P on that Interval.

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How to find the interval

① Find all the discontinuities of $p(x)$ and discontinuities of $q(x)$.

② put the equation on the standard form

ملاحظة: لا بد ان y' لا يزداد من y وانما.

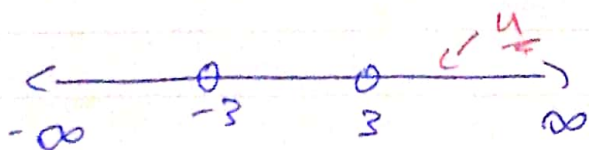
Ex: Find the largest interval in which a Unique solution is guaranteed to exist.

$$\text{I) } y'(x^2-4) + 2y = \frac{\ln|20-\sqrt{x}|}{x^2-4}, \quad y(4) = -3$$

Sol: $y' + \frac{2}{x^2-4}y = \frac{\ln|20-\sqrt{x}|}{x^2-4}$

$$p(x) = \frac{2}{x^2-4}$$

$$x^2-4=0 \Rightarrow x = \pm 2$$



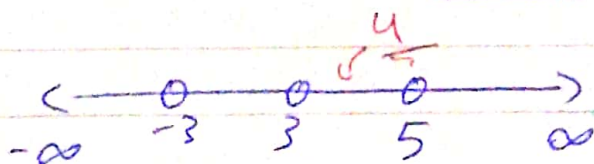
$\therefore (2, \infty)$

$$q(x) = \frac{\ln|20-\sqrt{x}|}{x^2-4}$$

$$x^2-4=0 \Rightarrow x = \pm 2$$

$$20-\sqrt{x} > 0$$

$$20-\sqrt{x}=0 \Rightarrow 20=\sqrt{x} \Rightarrow \boxed{x=400}$$



$\therefore (2, 400)$

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∴ The largest interval is:-

$$(3, \infty) \cap (3, 5) \Rightarrow \underline{(3, 5)}$$

2] $\tan x y' - \sec x \tan^2 x y = 0$, $y(0) = \pi$

Sol:- $(-\frac{\pi}{2}, \frac{\pi}{2})$

3] $(x-9)y' + \frac{2x}{\ln(x-5)} y = \frac{3}{x}$, $y(7) = 3$

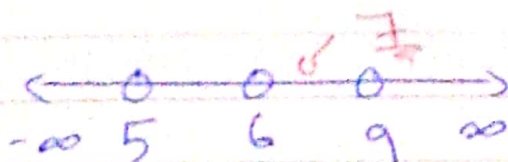
Sol:- $y' + \frac{2x}{(x-9)\ln(x-5)} y = \frac{3}{x(x-9)}$

$$p(x) = \frac{2x}{(x-9)\ln(x-5)}$$

$$x-9=0 \Rightarrow \boxed{x=9}$$

$$\ln(x-5)=0 \Rightarrow x-5=1 \Rightarrow \boxed{x=6}$$

$$x-5 > 0 \Rightarrow \boxed{x > 5}$$



$$\therefore (6, 9)$$

$$q(x) = \frac{3}{x(x-9)}$$

$$\boxed{x=0}$$

$$x-9=0 \Rightarrow \boxed{x=9}$$



$$\therefore (0, 9)$$

∴ The largest interval is:- $(6, 9) \cap (0, 9)$

$$= \underline{(6, 9)}$$

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chapter (2) :- First order D.E.

The D.E has the form $\rightarrow \frac{dy}{dx} = f(x, y)$

1] separable equation

• ملاحظة :- إذا قدرت فصل بين x و y بعد الحاصل الثاني
• بطريقة ال separable وهي الطريقة بعد الحاصل الثاني

standard form of separable equations:-

$$\rightarrow \frac{dy}{dx} = g(x) h(y)$$

• خطوات حل سوال ال separable :-

1] فصل x عن y

2] دمج الطرفين.

Ex- 1] $\frac{dy}{dx} = e^{x-3y}$

• ملاحظة :-

$$\frac{dy}{dx} = e^x \cdot e^{-3y} \rightarrow \frac{1}{e^{-3y}} dy = e^x \cdot dx \quad (\checkmark) \text{ sep.}$$

$$2] \frac{dy}{dx} = \frac{xy + y}{yx + x} \rightarrow \frac{dy}{dx} = \frac{y(x+1)}{x(y+1)} \rightarrow \frac{y+1}{y} dy = \frac{x+1}{x} \cdot dx$$

($\checkmark$) sep.

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Ex 3 - Solve the D.E $y^2 (1-x^2)^2 dy = \sin^{-1}(x) \cdot dx$
 $y(0) = 1$

sol $\int y^2 \cdot dy = \int \frac{\sin^{-1}(x)}{\sqrt{1-x^2}} \cdot dx$

* قدرت انفصال الـ y درجة والـ x درجة فصلها بالـ 3 -

$$\frac{1}{3} y^3 = \int \frac{\sin^{-1}(x) \cdot dx}{\sqrt{1-x^2}}$$

$$u = \sin^{-1}(x)$$

$$du = \frac{1}{\sqrt{1-x^2}} \cdot dx$$

$$\frac{1}{3} y^3 = \int \frac{u}{\sqrt{1-x^2}} \cdot \sqrt{1-x^2} du \quad \leftarrow \quad dx = \sqrt{1-x^2} du$$

$$\frac{1}{3} y^3 = \frac{u^2}{2} + C \rightarrow \frac{1}{3} y^3 = \frac{(\sin^{-1}(x))^2}{2} + C$$

$$y(0) = 1 \rightarrow \frac{1}{3} (1)^3 = \frac{(\sin^{-1}(0))^2}{2} + C$$

$$\frac{1}{3} = 0 + C \rightarrow \underline{C = \frac{1}{3}}$$

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Ex 3 - solve $y' = \frac{xy + 3x - y - 3}{xy - 2x + 4y - 8}$

Solve - $\frac{dy}{dx} = \frac{x(y+3) - (y+3)}{x(y-2) + 4(y-2)}$

$$\frac{dy}{dx} = \frac{(y+3)(x-1)}{(y-2)(x+4)} \rightarrow \int \frac{y-2}{y+3} dy = \int \frac{x-1}{x+4} dx$$

* نكسر د. درجة البسط أعلى أو يساوي درجة المقام قسمه طويلا

$$\begin{array}{r} 1 \\ y+3 \overline{) y-2} \\ \underline{y+3} \\ -5 \end{array} \rightarrow \int \left(1 + \frac{-5}{y+3} \right) \cdot dy$$

$$\begin{array}{r} 1 \\ x+4 \overline{) x-1} \\ \underline{x+4} \\ -5 \end{array} \rightarrow \int \left(1 + \frac{-5}{x+4} \right) \cdot dx$$

$$\int \left(1 + \frac{-5}{y+3} \right) \cdot dy = \int \left(1 + \frac{-5}{x+4} \right) \cdot dx$$

$$y - 5 \ln|y+3| = x - 5 \ln|x+4| + C$$

نكسر ←
 $\int \frac{\text{الباقى}}{\text{المقام}} + \text{الجزء 2}$

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Ex 3 - solve $(2+x)dy - (y+3)dx = 0$

Sol 3 - $(2+x)dy = (y+3)dx$

① نفصل . ② نكامل الطرفين

$$\int \frac{dy}{y+3} = \int \frac{dx}{2+x}$$

$$\Rightarrow \ln|y+3| = \ln|2+x| + C$$

$$|y+3| = e^{\ln|2+x|} \cdot e^C \rightarrow y = C|2+x| - 3 \text{ explicit soln}$$

$$y = -C|2+x| - 3 \text{ explicit soln}$$

Ex 3 - solve $\frac{dy}{dx} = \frac{xy + 3x + 2y + 6}{xy - 2y + 4x - 8}$

Sol 3 - $\frac{dy}{dx} = \frac{x(y+3) + 2(y+3)}{y(x-2) + 4(x-2)}$

$$\frac{dy}{dx} = \frac{(y+3)(x+2)}{(x-2)(y+4)} \rightarrow \int \frac{y+4}{y+3} dy = \int \frac{x+2}{x-2} dx$$

$\rightarrow y - \ln|y+3| = x + 4\ln|x-2| + C$ (Implicit solution).

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2] linear equations.

Standard form of linear equation:-

$$\rightarrow \frac{dy}{dx} + p(x)y = f(x)$$

• ملاحظہ کریں کہ یہ linear ہے اور $\frac{dy}{dx}$ کا درجہ 1 ہے۔

* Integrating factor (I.F) :-

$$M(x) = e^{\int p(x) \cdot dx}$$

• یہ linear ہے اور $\frac{dy}{dx}$ کا درجہ 1 ہے۔

① put the D.E on the standard form.

② Find I.F $\rightarrow M(x) = e^{\int p(x) \cdot dx}$

$$③ y = \frac{1}{M(x)} \left[C + \int M(x) \cdot f(x) dx \right]$$

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Ex 3 - $xy' - 4y = x^6 e^x, x > 0$

Sol 3 - $y' - \frac{4}{x}y = x^5 e^x \rightarrow f(x)$
 (1) do let $P(x)$

$$M(x) = e^{\int \frac{4}{x} dx} \rightarrow e^{-4 \ln(x)} \rightarrow M(x) = \frac{1}{x^4}$$

$$y = x^4 \left[C + \int x^{-4} x^5 e^x dx \right]$$

$$y = x^4 \left[C + \int x e^x dx \right]$$

$$y = x^4 \left[C + x e^x - e^x \right] \rightarrow y = Cx^4 + x^5 e^x - x^4 e^x$$

Ex 3 - solve $dy + 2y dx = e^{-2x} dx$

Sol 3 - $\frac{dy}{dx} + 2y = e^{-2x}$

$$M(x) = e^{\int 2 dx} \rightarrow M(x) = e^{2x}$$

$$y = \frac{1}{e^{2x}} \left[C + \int e^{2x} \cdot e^{-2x} dx \right]$$

$$y = \frac{1}{e^{2x}} [C + x]$$

$$y = C e^{-2x} + e^{-2x} x$$

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→ $x' + p(y)x = f(y)$ linear in x . -3 nolo nolo x

→ $y' + p(x)y = f(x)$ linear in y .

Ex 3 - solve $\frac{dy}{dx} = \frac{1}{e^y - x}$

Sol:- $\frac{dx}{dy} = e^y - x \rightarrow \frac{dx}{dy} + x = e^y$
linear in (x) .

$$M(y) = e^{\int 1 \cdot dy} \Rightarrow \underline{e^y}$$

$$x = \frac{1}{e^y} \left[C + \int e^y \cdot e^y \cdot dy \right]$$

$$x = e^{-y} \left[C + \frac{1}{2} e^y \right] \rightarrow x = C e^{-y} + \frac{1}{2} e^y$$

Ex 3 - solve $(x+2)^2 dy = (5 - 4y - 2xy) dx$

Sol - $(x+2)^2 \frac{dy}{dx} = 5 - 4y - 2xy$

$$(x+2)^2 y' + 4y + 2xy = 5 \rightarrow (x+2)^2 y' + (4+2x)y = 5$$

$$y' + \frac{2(x+2)}{(x+2)^2} y = \frac{5}{(x+2)^2}$$

$$y' + \frac{2}{x+2} y = \frac{5}{(x+2)^2}$$

$$\rightarrow M(x) = e^{\int \frac{2}{x+2} dx} = e^{2 \ln|x+2|} = e^{\ln(x+2)^2} = (x+2)^2$$

$$y = \frac{1}{(x+2)^2} \left[C + \int (x+2)^2 \cdot \frac{5}{(x+2)^2} dx \right]$$

$$y = (x+2)^{-2} [C + 5x] \Rightarrow y = C(x+2)^{-2} + 5x(x+2)^{-2}$$

Ex 3 - I f $M(x) = (x+2)^2$ is an I.F for the linear

D.E $\rightarrow y' + p(x)y = \frac{5}{(x+2)^2}$; find $p(x)$?

Sol $M(x) = (x+2)^2 \rightarrow e^{\int p(x) dx} = (x+2)^2$

$$\int p(x) dx = \ln(x+2)^2 \rightarrow \int p(x) dx = 2 \ln(x+2)$$

$\rightarrow$ اشتق الطرفين $\rightarrow p(x) = \frac{2(1)}{x+2} \Rightarrow p(x) = \frac{2}{x+2}$

Ex 3- IF $f'(t) = \frac{f^2(t) + 1}{t+1}$ with $f(0) = 0$;
find $f(2)$?!

Sol $\rightarrow$ $f' = \frac{f^2 + 1}{t+1} \Rightarrow \frac{df}{dt} = \frac{f^2 + 1}{t+1}$

$$\int \frac{df}{f^2 + 1} = \int \frac{dt}{t+1} \Rightarrow \tan^{-1}(f) = \ln(t+1) + C$$

$$\tan^{-1}(0) = \ln(0+1) + C$$

$$\underline{C=0}$$

$$\rightarrow \tan^{-1}(f(t)) = \ln|t+1| + 0 \Rightarrow \text{Now } f(2) =$$

$$\rightarrow f(2) = \tan^{-1}(f(2)) = \ln|2+1| \rightarrow \tan^{-1}(f(2)) = \ln(3)$$

$$f(2) = \tan(\ln(3)) \leftarrow \text{بعض كيب tan للجزءين}$$

H.w 3- Solve $\frac{dr}{d\theta} + r \sec \theta = \cos \theta$

3] Bernoulli equation.

Def:- A first order D.E that can be written in the form-

$$y' + Q(x)y = g(x)y^n$$

→ where $Q(x)$ and $g(x)$ are continuous functions on (a, b) and n is real number is said to be Bernoulli equation.

* To solve the Ber. eq convert to linear eq by substitution:-

$$V = y^{1-n} \quad (1)$$

$$V' = (1-n)y^{-n} \cdot y' \rightarrow y' = \frac{V'}{(1-n)y^{-n}}$$

→ Nil ekh jago ←

$$\frac{V}{(1-n)y^{-n}} + Q(x) \cdot y = g(x)y^n$$

$$\frac{V'}{1-n} + Q(x)y' \cdot y^{-n} = g(x)y^n \cdot y^{-n}$$

$$V' + Q(x)(1-n)y^{1-n} = (1-n)g(x) \quad (2)$$

$$\rightarrow V' + (1-n)Q(x)V = (1-n)g(x) \quad \text{linear in } V.$$

Ex 3 - Solve the D.E $y' + xy = xy^2$

Sol 3 - $n=2$, $v = y^{1-2} \rightarrow \underline{v = y^{-1}}$

$$v' + (1-n)\phi(x)v = (1-n)g(x)$$

$$v' - xv = (-1)x \rightarrow \underline{v' - xv = -x} \text{ linear in } v$$

$$\rightarrow \mu(x) = e^{\int -x \cdot dx} \Rightarrow \mu(x) = e^{-\frac{x^2}{2}}$$

$$v = \frac{1}{e^{-\frac{x^2}{2}}} \left[C + \int e^{-\frac{x^2}{2}} \cdot (-x) \cdot dx \right]$$

$$v = e^{\frac{x^2}{2}} \left[C + \int (-x) e^{-\frac{x^2}{2}} \cdot dx \right]$$

$$v = e^{\frac{x^2}{2}} \left[C + e^{-\frac{x^2}{2}} \right] \rightarrow v = Ce^{\frac{x^2}{2}} + 1$$

$$y^{-1} = Ce^{\frac{x^2}{2}} + 1 \rightarrow y = \frac{1}{Ce^{\frac{x^2}{2}} + 1}$$

Ex 2 - Solve $\frac{1}{\sqrt{y}} dy + x\sqrt{y} dx = 6x \cdot dx$

Sol 3 - $\frac{1}{\sqrt{y}} \cdot \frac{dy}{dx} + x\sqrt{y} = 6x \rightarrow \frac{dy}{dx} + xy = 6x\sqrt{y}$

$$y' + xy = 6xy^{\frac{1}{2}} \rightarrow \underline{n = \frac{1}{2}} \\ \underline{v = y^{1-\frac{1}{2}} = v = y^{\frac{1}{2}}}$$

$$\underline{v' + \frac{1}{2}x \cdot v = 3x}$$

linear in $v \rightarrow \mu(x) = \dots$

Ex 3. Solve $\frac{dy}{dx} = \frac{1}{\frac{x}{y} + yx^2}$, $X(1)=2$; find $X(3)$?

Sol 3. $\frac{dx}{dy} = \frac{x}{y} + yx^2 \rightarrow X' - \frac{1}{y}X = yx^2$ Ber in X
 $v = X^{1-n} \Rightarrow v = X^{-1}$
 $n = 2$

$$v' - \left(-\frac{1}{y}\right)v = (-1)y \rightarrow v' + \frac{1}{y}v = -y \text{ linear in } v.$$

$$\rightarrow \mu(y) = e^{\int \frac{1}{y} dy} \Rightarrow \mu(y) = y$$

$$v = \frac{1}{y} \left[C + \int y \cdot (-y) \cdot dy \right]$$

$$v = Cy^{-1} - \frac{y^2}{3} \rightarrow X^{-1} = Cy^{-1} - \frac{y^2}{3}$$

$$X = \frac{1}{Cy^{-1} - \frac{y^2}{3}} \rightarrow 2 = \frac{1}{C - \frac{1}{3}} \rightarrow \underline{C = \frac{5}{6}}$$

$$\rightarrow \text{Now, } X(3) = \frac{1}{\frac{5}{6} \left(\frac{1}{3}\right) - 3} \rightarrow \underline{X = \frac{-18}{11}}$$

- مقررته قبل الامتحان الرابعة لتوضيح بعض المفاهيم -

Ex 3:- $f(x,y) = xy^2 + \sin x + 2y$; find:-

$$\textcircled{1} \frac{\partial f}{\partial x} = y^2 + \cos x + 0$$

$$\textcircled{2} \frac{\partial f}{\partial y} = 2xy + 0 + 2$$

Ex 3:- $f(x,y) = e^{2xy} + \tan(xy) + 3xy$; find:-

$$\textcircled{1} \frac{\partial f}{\partial x} = 2y e^{2xy} + y \cdot \sec^2 xy + 3y.$$

$$\textcircled{2} \frac{\partial f}{\partial y} = 2x e^{2xy} + x \cdot \sec^2 xy + 3x.$$

Ex 3:- 1] $\int (x + xy) \cdot dx = \frac{x^2}{2} + \frac{x^2}{2} y + \underline{g(y)}$

2] $\int (e^{xy} + \cos x) \cdot dy = \frac{e^{xy}}{x} + y \cos x + \underline{g(x)}$

4] Exact equations.

* مثال ٢٠ حلّ هذا المعادلة -
اشتق جانباً إلى يسار المعادلة ← $xy^2 + y \sin x = 10$

$$2xy \cdot y' + y^2 + y \cos x + \sin x \cdot y' = 0$$

$$2xy \frac{dy}{dx} + y^2 + y \cos x + \sin x \cdot \frac{dy}{dx} = 0$$

$$2xy \, dy + y^2 \cdot dx + y \cos x \cdot dx + \sin x \cdot dy = 0$$

$$(y^2 + y \cos x) \cdot dx + (2xy + \sin x) \cdot dy = 0$$

$$\frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} \cdot dy = 0$$

Note 3:- If $f(x, y) = \text{Constant } (C)$; Then:-

$$\frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} \cdot dy = 0$$

→ The D.E $M(x, y) \cdot dx + N(x, y) \cdot dy = 0$
is said to be exact if:-

$$\rightarrow \left| \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \right|$$

$\downarrow \qquad \qquad \downarrow$
 $M_y = N_x$

Ex 3 Solve the D.E:-

$$1] \underbrace{(2xy + \sin x)}_M \cdot dx + \underbrace{(x^2y + \cos x)}_N \cdot dy = 0$$

Sol:- $M_y = 2x$, $N_x = 2xy - \sin x$

$\times M_y \neq N_x \rightarrow \text{Not Exact.}$

$$2] \underbrace{(2xy + \cos x)}_M \cdot dx + \underbrace{(x^2 + \cos y)}_N \cdot dy = 0$$

Sol:- $M_y = 2x$, $N_x = 2x$

$\times M_y = N_x \rightarrow \text{Exact}$

ملاحظة: لهذه الحالة بأنها Exact نبينها بالأمثلة الآتية:-

$$\frac{\partial f}{\partial x} = 2xy + \cos x$$

① أول خطوة هي اختيار أحد المتكاملات الجزئية إما M أو N

$$\int \frac{\partial f}{\partial x} \cdot dx = \int (2xy + \cos x) \cdot dx$$

$$f(x,y) = x^2y + \sin x + g(y)$$

② الخطوة الثانية هي إيجاد $g(y)$

$$\frac{\partial f}{\partial y} = x^2 + 0 + g'(y)$$

③ الخطوة الثالثة نشتق المعادلة التي وضعناها بمقدورنا وذلك لإيجاد قيمة $g(y)$ لكن انبثقت المشتق بالنسبة لـ y هي المرة

$$\cancel{x^2} + \cos y = \cancel{x^2} + g'(y)$$

④ الخطوة الرابعة تقريظ المشتق $\frac{\partial f}{\partial y}$ من المعادلة الأخيرة بطالع حثية اختيار:

$$g'(y) = \cos y \rightarrow g(y) = \int \cos y \cdot dy$$

$$g(y) = \sin y$$

⑤ الخطوة الخامسة بعد الإختبار، يمكننا التحقق من حسابنا الأخير على حثية $g(y)$ بكون المعادلة (C).

$$f(x, y) = x^2 y + \sin x + g(y)$$

$$\rightarrow C = x^2 y + \sin x + \sin y$$

⑥ الخطوة السادسة نرجع للمعادلة التي وضعناها بمقدورنا ونعوين فيها قيمة $g(y)$ ونعوين C بدل $f(x, y)$.

$$f(x, y) = C \quad \# \text{ تذكير}$$

* يمكن أحياناً التعامل بطريقة ثانية لكن بنفس الخطوات 6 و 7
 بإختيار الحد الثاني من المعادلة $\rightarrow \frac{\partial f}{\partial y} = N$

$$\frac{\partial f}{\partial y} = x^2 + \cos y$$

$$\int \frac{\partial f}{\partial y} dy = \int (x^2 + \cos y) dy$$

$f(x, y) = yx^2 + \sin y + g(x)$ $\rightarrow$ انبه $g(x)$ بدل $g(y)$.

$$\frac{\partial f}{\partial x} = 2xy + g'(x)$$

$$2xy + \cos x = 2xy + g'(x)$$

$$\int g'(x) = \int \cos x \rightarrow g(x) = \sin x$$

$$C = yx^2 + \sin y + \sin x$$

Ex:- Solve $(\cos x \sin x - xy^2) + (y - yx^2)y' = 0$

Sol:- $\left[(\cos x \sin x - xy^2) + (y - yx^2) \frac{dy}{dx} = 0 \right] \times dx$

$$(\cos x \sin x - xy^2)dx + (y - yx^2)dy = 0$$

$$\frac{\partial f}{\partial x}$$

$$\frac{\partial f}{\partial y}$$

$$M_y = -2xy, \quad N_x = -2xy \rightarrow M_y = N_x \text{ Exact}$$

$$\frac{\partial f}{\partial y} = \int (y - yx^2) \cdot dy$$

$$f(x, y) = \frac{y^2}{2} - \frac{y^2}{2} x^2 + g(x)$$

$$\frac{\partial f}{\partial x} = y^2 x + g'(x)$$

$$\cancel{\cos x \sin x} - \cancel{x y^2} = -\cancel{y^2 x} + g'(x)$$

$$\int g'(x) = \int \cos x \sin x \cdot dx$$

$$g(x) = \int \frac{1}{2} \sin 2x \cdot dx \rightarrow g(x) = -\frac{1}{4} \cos 2x$$

$$C = \frac{y^2}{2} - \frac{y^2}{2} x^2 - \frac{1}{4} \cos 2x$$

Ex 3 - If the D.E $M(x, y) dx + (x^2 y + \sin x) \cdot dy = 0$ is exact; find $M(x, y)$?

Sol:- $M_y = N_x \leftarrow \text{exact}$

$$M_y = 2xy + \cos x \rightarrow \int M_y = \int (2xy + \cos x) \cdot dy$$

$$M = xy^2 + y \cos x + g(x)$$

$$\rightarrow M(x, y) = xy^2 + y \cos x + g(x)$$

Ex 3- If the D.E $(ay^3 + bxy^4 - 2x) \cdot dx + (3xy^2 + 20x^2y^3) \cdot dy = 0$ is exact; find the value of a and b ?

sol 3- $M_y = N_x \rightarrow 3ay^2 + 4bxy^3 = 3y^2 + 40xy^3$

$3a = 3 \Rightarrow \boxed{a=1}$

$4b = 40 \Rightarrow \boxed{b=10}$

* $\leftarrow$ $\frac{M_y - N_x}{N}$ or $\frac{N_x - M_y}{M}$ *

Ex 3- If the D.E $(xy^3 + \ln y) \cdot dx + (x + N(x,y)) \cdot dy = 0$ is exact; find the function $N(x,y)$?

sol 3- Exact $\Rightarrow 3xy^2 + \frac{1}{y} = 1 + N_x$

$\int N_x = \int (3xy^2 + \frac{1}{y} - 1) \cdot dx$

$\rightarrow N(x,y) = \frac{3}{2} x^2 y^2 + \frac{x}{y} - x + g(y)$

Integrating factors for non-exact equation.

1] If $M(x,y) \cdot dx + N(x,y) \cdot dy = 0 \rightarrow$ This is not exact

2] If $\frac{M_y - N_x}{N}$ is function on x alone; then

the I.F $\rightarrow \mu(x) = e^{\int \frac{M_y - N_x}{N} \cdot dx}$

3] If $\frac{N_x - M_y}{M}$ is a function on y alone; then

the I.F $\rightarrow \mu(y) = e^{\int \frac{N_x - M_y}{M} \cdot dy}$

Ex 3- Find the I.F for the following non-exact equation:-

$$\textcircled{1} \underbrace{(x+y)}_M \cdot dx + x \underbrace{\ln x}_N \cdot dy = 0, \quad x > 0$$

so $M_y = 1, \quad N_x = x \cdot \frac{1}{x} + \ln x = 1 + \ln x$

$\Rightarrow$ non-exact $M_y \neq N_x$

$$\text{so, } \frac{M_y - N_x}{N} = \frac{1 - (1 + \ln x)}{x \ln x} = \frac{-\ln x}{x \ln x} = \frac{-1}{x}$$

$$M(x) = e^{\int \frac{-1}{x} \cdot dx} \Rightarrow e^{-\ln x} = e^{\ln x^{-1}} \Rightarrow \underline{M(x) = x^{-1}}$$

هو المطلوب فقط الـ I.F ان كان ايسا بسا بسا لا بد ان يكون

$$\frac{1}{x} (x+y) \cdot dx + \frac{1}{x} (x \ln x) \cdot dy = 0$$

$$(1 + \frac{1}{x}) \cdot dx + \ln x \cdot dy = 0$$

$$M_y = \frac{1}{x}, \quad N_x = \frac{1}{x} \Rightarrow M_y = N_x \text{ exact } (\checkmark).$$

$$\textcircled{2} \underbrace{y}_{M} \cdot dx + \underbrace{(2x - ye^y)}_N \cdot dy = 0$$

Sol: $M_y = 1, N_x = 2$

$$\frac{M_y - N_x}{N} = \frac{1 - 2}{2x - ye^y} = \frac{-1}{2x - ye^y} \quad \times$$

لازم تكون $\frac{M_y - N_x}{N}$ دالة في y فقط $\times$

$$\frac{N_x - M_y}{M} = \frac{2 - 1}{y} = \frac{1}{y} \Rightarrow M(y) = e^{\int \frac{1}{y} \cdot dy}$$

$M(y) = y$

Ex: The ODE $(y^2 + x) \cdot dx + f(x) \cdot dy = 0$, becomes exact by I.F $M(x) = x$, find $f(x)$?

Sol: $M(x) = x \rightarrow x(y^2 + x) \cdot dx + x f(x) \cdot dy = 0$

$$(xy^2 + x^2) \cdot dx + x f(x) \cdot dy = 0$$

$$M_y = 2xy, N_x = (x f(x))'$$

$$\Rightarrow \int 2xy \cdot dx = \int (x f(x))' \rightarrow x^2 y = x f(x)$$

$$\downarrow$$

$$\underline{f(x) = xy + cx^{-1}}$$

Ex 3 - If $M_y - N_x = X \cdot N(x, y)$, and if

$$\frac{M_y - N_x}{N} = e^x, \text{ solve the ODE}$$

$$(M_y + N_x) \cdot dx + (\ln x) \cdot dy = 0 \quad ?!$$

Sol 3 - $M_y - N_x = X \cdot N(x, y)$

$$(M_y - N_x)(M_y + N_x) = X N(x, y)$$

$$M_y + N_x = \frac{X \cdot N(x, y)}{M_y - N_x} = e^{-x} \cdot X$$

$$(M_y + N_x) \cdot dx + \ln x \cdot dy = 0$$

$$X e^{-x} \cdot dx + \ln x \cdot dy = 0 \Rightarrow \int \frac{X e^{-x}}{\ln x} \cdot dx = \int -dy$$

$$y = - \int \frac{X e^{-x}}{\ln x} \cdot dx$$

Ex 3 - If $XM + yN = 0$, solve the DE

$$M(x, y) \cdot dx + N(x, y) \cdot dy = 0$$

Sol 3 - $M \cdot dx = -N \cdot dy \rightarrow \frac{M}{N} = -\frac{dy}{dx} \rightarrow \frac{y}{x} = \frac{dy}{dx}$

$$\frac{dx}{x} = \frac{dy}{y} \rightarrow \ln x + C = \ln y$$

H.W 3 - Find an I.F for the D.E

$$\left(4\left(\frac{x^3}{y^4}\right) + \frac{3}{y}\right) \cdot dx + \left(3\left(\frac{x}{y^2}\right) + 4y\right) \cdot dy = 0$$

Ex 3 - Find m and n s.t. $M(x,y) = x^m \cdot y^n$ that is makes the D.E $\rightarrow M(x,y) = x^2y$

$$(4xy^2 + 3y) \cdot dx + (3x^2y + 2x) \cdot dy = 0 \text{ is exact}$$

So 3 - $x^m \cdot y^n (4xy^2 + 3y) \cdot dx + x^m \cdot y^n (3x^2y + 2x) \cdot dy = 0$

$$\left(4x^{m+1} \cdot y^{n+2} + 3x^m y^{n+1}\right) \cdot dx + \left(3x^{m+2} \cdot y^{n+1} + 2x^{m+1} \cdot y^n\right) \cdot dy = 0$$

M

$$M_y = N_x$$

N

$$4(n+2) \underline{x^{m+1} \cdot y^{n+1}} + 3(n+1) \underline{x^m \cdot y^n} = 3(m+2) \underline{x^{m+1} \cdot y^{n+1}} + 2(m+1) \underline{x^m \cdot y^n}$$

$$3(n+1) = 2(m+1) \text{ --- (1)}$$

$$4(n+2) = 3(m+2) \text{ --- (2)}$$

$$\boxed{n=1}, \boxed{m=2}$$

4] Homogeneous equation.

Def³ - A function $G(x, y)$ is called Homogeneous of order n if $G(tx, ty) = t^n G(x, y)$.

Ex³ ① $G(x, y) = x^2 + y^2$

$$G(tx, ty) = t^2 x^2 + t^2 y^2 \rightarrow t^2 (x^2 + y^2) \text{ Homo. of order } \underline{2}.$$

② $G(x, y) = x^2 + xy$

$$G(tx, ty) = t^2 (x^2 + xy) \text{ Homo. of order } \underline{2}.$$

③ $G(x, y) = x^2 + xy^2$

$$G(tx, ty) = t^2 x^2 + t^3 xy^2 \text{ non-Homo}$$

Def³ - A differential eq. $M(x, y) \cdot dx + N(x, y) \cdot dy = 0$ is said to be Homogeneous if the function $M(x, y)$ and $N(x, y)$ are Homo of the same order.

Ex 3-1 $(x^2 + y^2) \cdot dx - (xy + y^2) \cdot dy = 0$

Sol 3- $(x^2 + y^2) \cdot dx = (xy + y^2) \cdot dy$

$$\frac{x^2 + y^2}{xy + y^2} = \frac{dy}{dx} \leadsto \text{Homo (same order)}$$

$$\frac{dy}{dx} = \frac{\frac{x^2}{x^2} + \frac{y^2}{x^2}}{\frac{xy}{x^2} + \frac{y^2}{x^2}} \Rightarrow \frac{dy}{dx} = \frac{1 + \left(\frac{y}{x}\right)^2}{\frac{y}{x} + \left(\frac{y}{x}\right)^2}$$

$$= \frac{1 + v^2}{v + v^2}$$

$$v = \frac{y}{x}$$

2) $y e^{\frac{y}{x}} \cdot dx - y \cos \frac{y}{x} \cdot dy = 0$

Sol 3- $\frac{y}{x} e^{\frac{y}{x}} \cdot dx - \frac{y}{x} \cos \frac{y}{x} \cdot dy = 0 \leadsto \text{Homo}$

$$v = \frac{y}{x}$$

3) $y [\ln y - \ln x] y' = 0$

Sol 3- $\frac{y}{x} [\ln \frac{y}{x}] = 0 \leadsto \text{Homo}$

$$v = \frac{y}{x}$$

How to solve the Homo. equation?!

→ put $v = \frac{y}{x}$ and then transform the eq. to sep, linear, ----

Ex3- solve $\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}$?!

Sol:

• x^2 de + JS eq.

$$\frac{dy}{dx} = \frac{\frac{x^2}{x^2} + \frac{xy}{x^2} + \frac{y^2}{x^2}}{\frac{x^2}{x^2}}$$

$$\frac{dy}{dx} = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2 \rightarrow v = \frac{y}{x} \Rightarrow y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

orig $\rightarrow v + x \frac{dv}{dx} = 1 + v + v^2$

$$x \frac{dv}{dx} = 1 + v^2 \quad \text{sep}$$

$$\int \frac{dv}{1+v^2} = \int \frac{dx}{x} \Rightarrow \tan^{-1} v = \ln|x| + C$$
$$\tan^{-1}\left(\frac{y}{x}\right) = \ln|x| + C$$
$$\frac{y}{x} = \tan(\ln|x| + C).$$

Ex. 2 solve $\frac{dy}{dx} = \frac{2y + \sqrt{x^2 - y^2}}{2x}$, $x > 0$?!

Sol. 2. $\frac{dy}{dx} = \frac{\frac{2y}{x} + \sqrt{\frac{x^2}{x^2} + \frac{y^2}{x^2}}}{\frac{2x}{x}}$

$\frac{dy}{dx} = \frac{y}{x} + \frac{1}{2} \sqrt{1 - \left(\frac{y}{x}\right)^2}$ $\left| v = \frac{y}{x} \right| \rightarrow y = vx$

$\frac{dy}{dx} = v + x \frac{dv}{dx}$

نزلنا في المتغير v $\rightarrow v + x \frac{dv}{dx} = v + \frac{1}{2} \sqrt{1 - v^2}$

$x \frac{dv}{dx} = \frac{1}{2} \sqrt{1 - v^2}$ sep

$\int \frac{2 dv}{\sqrt{1 - v^2}} = \int \frac{dx}{x} \rightarrow 2 \sin^{-1}(v) = \ln(x) + C$

$\sin^{-1}(v) = \frac{1}{2} \ln(x) + C$

$v = \sin\left(\frac{1}{2} \ln(x) + C\right)$

$\frac{y}{x} = \sin\left(\frac{1}{2} \ln(x) + C\right)$

Ex 3 - Solve $xy' = y \ln \frac{y}{x}$

Sol $\rightarrow \frac{dy}{dx} = \frac{y}{x} \ln \frac{y}{x}$

$\boxed{v = \frac{y}{x}} \rightarrow y = v + x \cdot \frac{dv}{dx}$

$$v + x \frac{dv}{dx} = v \ln v \rightarrow x \frac{dv}{dx} = v \ln v - v$$

$$\int \frac{dv}{v(\ln v - 1)} = \int \frac{dx}{x} \quad \int \frac{dv}{v \ln v - v} = \int \frac{dx}{x}$$

$$\ln(\ln v - 1) = \ln x + C$$

$$\ln\left(\ln \frac{y}{x} - 1\right) = \ln x + C$$

H.W :- ① $y' = \frac{y}{x} - \frac{x}{y}$

② $x^2 y' - xy = (x^2 + y^2) \cdot \tan^{-1}\left(\frac{y}{x}\right)$

③ $2x^3 y \cdot dx + (x^4 + y^4) \cdot dy = 0$

Ex 3. solve $\frac{dy}{dx} = 2\frac{y}{x} + \cos \frac{y}{x^2}$

Sol: $v = \frac{y}{x^2} \Rightarrow y = vx^2 \Rightarrow y' = 2vx + x^2 \frac{dv}{dx}$

$$2vx + x^2 \frac{dv}{dx} = \frac{2vx^2}{x} + \cos v$$

$$\cancel{2vx} + x^2 \frac{dv}{dx} = \cancel{2vx} + \cos v$$

$$x^2 \cdot \frac{dv}{dx} = \cos v \rightarrow \int \frac{dv}{\cos v} = \int dx x^{-2}$$

$$\ln |\sec v + \tan v| = -\frac{1}{x} + C$$

$$\ln \left| \sec \frac{y}{x^2} + \tan \frac{y}{x^2} \right| = -\frac{1}{x} + C$$

* Equation of the forms $\frac{dy}{dx} = G(ax+by+c)$

Ex³- solve $\frac{dy}{dx} = 2 + \sqrt{y-2x+3}$

Sol³- $|v = y-2x+3| \rightarrow$ لو حالة خاصة بتعرفن الي ذمت
الاجزاء مش شوا ذمت الجبر لازم تكون على الدمينه الي فوق

$$\frac{dv}{dx} = \frac{dy}{dx} - 2 \Rightarrow \left| \frac{dy}{dx} = \frac{dv}{dx} + 2 \right|$$

$\rightarrow$ في الدمينه v الانشأه $\frac{dv}{dx} + 2 = 2 + \sqrt{v}$

$$\frac{dv}{dx} = \sqrt{v} \rightarrow \int \frac{dv}{\sqrt{v}} = \int dx \rightarrow 2\sqrt{v} = x + C$$

$$2\sqrt{y-2x+3} = x + C$$

Ex³- solve $\frac{dy}{dx} = \sin(x-y)$

Sol³- $|v = x-y| \rightarrow \frac{dv}{dx} = 1 - \frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx} = 1 - \frac{dv}{dx} \xrightarrow{\text{في الدمينه انشأه}} 1 - \frac{dv}{dx} = \sin v$$

$$\frac{dv}{dx} = 1 - \sin v \Rightarrow \int \frac{dv}{1 - \sin v} = \int dx$$

• ما هي الطريقة المناسبة - 3

* what is the suitable substitute

Ex:- what is the suitable substitute that transform this equations:-

$$① (2x+2y) \cdot dx + (4x+4y+8) \cdot dy = 0$$

$$(2x+2y) \cdot dx + 2(2x+2y+4) \cdot dy = 0$$

$$\boxed{v = 2x+2y}$$

$$② (2x+2y+2) \cdot dx + (4x+4y+8) \cdot dy = 0$$

$$(2x+2y+2) \cdot dx + 2(2x+2y+2+2) \cdot dy = 0$$

$$\boxed{v = 2x+2y+2}$$

$$③ \frac{dy}{dx} = y-x-1 + (x-y+2)^{-1}$$

$$\frac{dy}{dx} = -(x-y-1) + (x+y+1+1)^{-1}$$

$$\boxed{v = x-y+1}$$

H.W. 3 - If $v \mapsto \frac{y}{x}$ reduce the Hom. eq

$M(x,y) \cdot dx + 2N(x,y) \cdot dy = 0$ to the form-

$$\frac{\frac{dx}{x} + 2N(1,v) \cdot dy}{M(1,v) + 2vN(1,v)} = 0 \quad ?!$$

* مراجعة سريعة لجميع المسائل :-

① $f(x) \cdot dx = g(y) \cdot dy \rightarrow \text{sep.}$

② $y' + p(x)y = f(x) \rightarrow \text{linear.}$

$$M(x) = e^{\int p(x) \cdot dx} \leadsto y = \frac{1}{M(x)} \left[C + \int M(x) \cdot f(x) \cdot dx \right]$$

③ $y' + p(x)y = g(x)y^n \rightarrow \text{Ber.}$

$$v = y^{1-n}$$

④ Exact. eq $M(x,y) \cdot dx + N(x,y) \cdot dy = 0$

$$M_y = N_x$$
$$M(x) = e^{\int \frac{M_y - N_x}{N}} \quad M(y) = e^{\int \frac{N_x - M_y}{M}}$$

⑤ $M(x,y) \cdot dx + N(x,y) \cdot dy = 0 \rightarrow \text{Homo.}$

M, N are Homa and same order.

⑥ equation $(ax + by + c).$

Examples:-

$$\textcircled{1} \frac{dy}{dx} = \frac{x-y}{x} \quad \text{Homo} \textcircled{1}$$

← زمره ضرب ضرب بتارلی 3-

$$(x-y) \cdot dx = x \cdot dy$$

$$(x-y) \cdot dx - x \cdot dy = 0$$

$$(x-y) \cdot dx + (-x) \cdot dy = 0$$

$$M_y = -1, N_x = -1 \rightarrow \text{Exact} \textcircled{2}$$

* یعنی بکن نقطه المارده القاطبة بالکتر من طرقة *

$$y' = \frac{x-y}{x} \rightarrow xy' + y = x \rightarrow y' + \frac{1}{x}y = 1 \rightarrow \text{linear} \textcircled{3}$$

$$\textcircled{2} \frac{dy}{dx} = \frac{1}{y-x}$$

Sol:- $\frac{dx}{dy} = y-x \rightarrow x' + x = y$ linear in x.

$$\textcircled{3} \frac{dy}{dx} = \frac{y^2 + y}{x^2 + x}$$

Sol:- $(x^2 + x) \cdot dy = (y^2 + y) \cdot dx$

$$\frac{dy}{y^2 + y} = \frac{dx}{x^2 + x} \quad \text{sep}$$

$$\textcircled{4} \frac{dx}{dy} = \frac{1}{5y + y^2}$$

Sol:- $\int dx = \int \frac{dy}{5y + y^2}$

sep

CH 3 3- linear differential equation of higher order.

* A linear-nth-order D.E of the form-

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = g(x)$$

is said to be **Homo**, where $a_n \neq 0$ is said to be **non-Homo**.

* Examples:- 1] $y'' + xy' = x+1$ **non-Homo**.

2] $y^{(4)} + (\sin x)y'' = 0$ **Homo**.

3] $xy'' + xy - 3 = 0$ **non-Homo**.

■ Thrm 3- Superposition principle-Homo. eq.

Let $y_1, y_2, \dots, y_n$ be a solution of the Homo nth order D.E:-

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_0y = 0.$$

→ Then the linear combination:-

$$y = C_1y_1 + C_2y_2 + C_3y_3 + \dots + C_ny_n \quad \text{also is a solution.}$$

Ex-3 - If $y_1 = e^{2x}$, $y_2 = e^{-2x}$ is a solution for D.E
 $y'' - 4y = 0$

Sol:- $y_1' = 2e^{2x}$, $y_1'' = 4e^{2x}$

$$y'' - 4y = 0 \rightarrow \text{Ans} \rightarrow 4e^{2x} - 4(e^{2x}) = 0 \checkmark$$

* لتفرض ان $y_1 = 10e^{2x}$ -310

$$y_1' = 20e^{2x}, \quad y_1'' = 40e^{2x}$$

$$y'' - 4y = 0 \rightarrow \text{Ans} \rightarrow 410e^{2x} - 4(10e^{2x}) = 0 \checkmark$$

2) $\text{Constant } \gamma$ يعني لو هزيت الامتداد بتبع المعادلة التفاضلية γ يعني كل المعادلة التفاضلية.

■ Def 3- Wronskian:

→ Suppose each of the function $y_1(x), y_2(x), \dots, y_n(x)$ possesses at least $n-1$ derivative. The determinant

$$W(y_1, y_2, \dots, y_n) = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}$$

-3 Wronskian -3 دالة ورونسكيان

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ determinant. } \text{دeterminate. } \text{دeterminate}$$

$$\downarrow$$

$$|A| = ad - cb$$

$$\det(A) = ad - cb$$

Ex 3. $A = \begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix}$

So 3. $|A| = 1(-3) - (2)(-1) = -3 + 2 = -1$

Ex 3. $\begin{vmatrix} e^{2x} & e^{-2x} \\ e^x & e^{3x} \end{vmatrix} = e^{2x}(e^{3x}) - e^x(e^{-2x})$

$$= e^{5x} - e^{-x}$$

-3 دالة ورونسكيان -3 دالة ورونسكيان

$$A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$|A| = a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$a(ei - hf) - b(di - gf) + c(dh - ge)$$

$$\text{Ex 3: } \begin{vmatrix} -1 & 2 & 0 \\ 3 & 4 & 2 \\ -1 & 2 & 7 \end{vmatrix}$$

$$\text{Sol: } -1 \begin{vmatrix} 4 & 2 \\ 2 & 7 \end{vmatrix} - 2 \begin{vmatrix} 3 & 2 \\ -1 & 7 \end{vmatrix} + 0$$

$$= -1(28-4) - 2(21+2) = -24-46 = \boxed{-70}$$

$$\text{Ex 3: } W(e^{2x}, e^{-3x})(0)$$

$$= \begin{vmatrix} e^{2x} & e^{-3x} \\ 2e^{2x} & -3e^{-3x} \end{vmatrix} = -3e^{-3x} \cdot e^{2x} - 2e^{2x} \cdot e^{-3x} \\ = -3e^{-x} - 2e^{-x} = -5e^{-x}$$

$$W(e^{2x}, e^{-3x})(0) = -5e^0 = \boxed{-5}$$

$$\text{Ex 3: Find } W(\sin x, \cos x)\left(\frac{\pi}{2}\right) ?!$$

$$\text{Sol: } \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x \\ = -1(\sin^2 x + \cos^2 x) = \boxed{-1}$$

Ex: If $w(e^x, y_2(x)) = 3$, find $y_2(x)$?

Sol:- $\begin{vmatrix} e^x & y_2 \\ e^x & y_2' \end{vmatrix} = 3 \Rightarrow e^x y_2' - e^x y_2 = 3$

$y_2' - y_2 = 3e^{-x}$ $\rightarrow$ linear.

$m(x) = e^{\int -1 \cdot dx} \Rightarrow m(x) = e^{-x}$

$y = e^x \left[c + \int e^{-x} (3e^{-x}) \cdot dx \right]$

$y = e^x \left[c + \frac{-3}{2} e^{-2x} \right] \Rightarrow y = \underline{\underline{ce^x - \frac{3}{2} e^{-x}}}$

Thrm Abel's Theorem.

If y_1 and y_2 are solution of the D.E

$$y'' + p(x)y' + q(x)y = 0$$

where p and q are continuous on I ; Then the Wronskian $W(y_1, y_2)(x)$ is given by:-

$$W(y_1, y_2)(x) = C \cdot e^{-\int p(x) \cdot dx}$$

→ where C is constant that depends on y_1 and y_2 , but not on x .

Ex:- Find the Wronskian for the following D.E

$$1] \quad xy'' + 2y' + xy = 0, \quad x > 0$$

Sol:- $y'' + \frac{2}{x}y' + y = 0$

$$W(y_1, y_2)(x) = C \cdot e^{-\int \frac{2}{x} \cdot dx} = C \cdot e^{-2\ln(x)} = C \cdot x^{-2}$$

$$2] \quad y''' - y'' - y' + 3y = 0$$

Sol:- $W(y_1, y_2)(x) = C \cdot e^{-\int -1 \cdot dx} \rightarrow C \cdot e^x$

* ملاحظة مهمة :- إذاً قيمة $p(x)$ تكون ثاني أو أعلى رتبة في المعادلة التفاضلية، ومن هنا نستنتج أن المعادلة رتبة في المعادلة وتكون معامل الرتبة التي أقل درجة مباشرة.

Ex:- For $x^2 y'' - 2y' + (3+x)y = 0$; if $x > 0$

$$w(y_1, y_2)(2) = 3, \text{ find } w(y_1, y_2)(5)$$

sol:- $y'' - \frac{2}{x^2} y' + \frac{(3+x)}{x^2} y = 0$

$$w(y_1, y_2)(x) = C \cdot e^{-\int \frac{2}{x^2}} = C \cdot e^{\frac{2}{x}}$$

$$w(y_1, y_2)(2) = 3 \rightarrow 3 = C \cdot e^{\frac{2}{2}} \Rightarrow 3 = C \cdot e^1$$

$$\underline{C = 3e^{-1}}$$

$$w(y_1, y_2)(x) = 3e^{-1} \cdot e^{\frac{2}{x}}$$

$$\text{So, } w(y_1, y_2)(5) = 3e^{-1} \cdot e^{\frac{2}{5}}$$

$$w(y_1, y_2)(5) = 3 \cdot e^{\frac{3}{5}}$$

Ex:- If $w(f(x), g(x))$ is the wronskian of f and g , and if $u = 2f - g$, $v = f + 2g$

find $w(u, v)(x)$?

Ex 1: $W(f, g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g$

$$W(u, v)(x) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} 2f-g & f+2g \\ 2f'-g' & f'+2g' \end{vmatrix}$$

$$= (2f-g)(f'+2g') - (2f'-g')(f+2g)$$

$$= \cancel{2ff'} + 4fg' - g f' - \cancel{2gg'} - (\cancel{2ff'} + 4fg' - \cancel{2gg'} - g' f)$$

$$= 4fg' - g f' - 4fg' + g' f$$

$$= 5fg' - 5g f' = 5(fg' - g f') = 5 \cdot W(f, g)$$

→ $W(u, v)(x) = 5 \cdot W(f, g).$

Thm We say that the set of solution

$y_1, y_2, y_3, \dots$ is linearly independent on interval I .

if $W(y_1, y_2, \dots, y_n) \neq 0$ and is said to be fundamental set of solution.

* If $y_1(x), y_2(x)$ are linearly dependent on I , then $W(y_1, y_2)(x) = 0$ for all x .

If $W(y_1, y_2) = 0$, does not mean that the functions are linearly ~~in~~ dependent.

Ex 3 - show that $y_1 = x^{\frac{1}{2}}, y_2 = x^{-1}$ are fundamental set of solution (linearly independent) of $2x^2 y'' + 3xy' - y = 0$

Sol 3. $W(x^{\frac{1}{2}}, x^{-1})(x) = \begin{vmatrix} x^{\frac{1}{2}} & x^{-1} \\ \frac{1}{2\sqrt{x}} & -\frac{1}{x^2} \end{vmatrix} = \frac{-x^{\frac{1}{2}}}{x^2} - \frac{x^{-1}}{2\sqrt{x}}$

$$= -x^{-\frac{3}{2}} - \frac{1}{2} x^{-\frac{3}{2}} = -\frac{3}{2} x^{-\frac{3}{2}}$$

Ex 3 - $W(e^{2x}, e^{-2x})(x) = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -2 - 2 = -4$

$-4 \neq 0$ $\rightarrow e^{2x}, e^{-2x}$ are linearly independent for all x .

* Reduction of order;

If we have a homogeneous second-order D.E
 $y'' + p(x)y' + q(x)y = 0$, if y_1 is the first
solution, y_2 is given by:-

$$y_2(x) = y_1 \int \frac{e^{-\int p(x) \cdot dx}}{(y_1)^2} \cdot dx$$

Ex 3 If $y_1 = e^x$ is a solution for:-

$xy'' - (x+1)y' + \frac{1}{x}y = 0$, find the other
solution?!

Sol:- $y'' - \frac{(x+1)}{x} \cdot y' + \frac{1}{x^2} y = 0$

$$y_2 = e^x \int \frac{e^{-\int \frac{(x+1)}{x} \cdot dx}}{e^{2x}}$$

$$y_2 = e^x \int \frac{e^x \cdot e^{-\ln x}}{e^{2x}} = e^x \int \frac{x}{e^x} \cdot dx$$

$$= e^x \int x e^{-x} \cdot dx = e^x [-x e^{-x} - e^{-x}] = -x - 1$$

* Homogeneous Linear eq. with constant coefficient.

• second-order eq $\rightarrow ay'' + by' + cy = 0$

$$ay''(x) + by'(x) + cy(x) = 0$$

* If we try to find a solution of the form:-

$y = e^{rx}$ $\rightarrow$ After substitution on the eq

$$y' = r e^{rx}, \quad y'' = r^2 e^{rx}$$

$$\rightarrow ar^2 e^{rx} + br e^{rx} + ce^{rx} = 0$$

$$\Rightarrow e^{rx} [ar^2 + br + c] = 0, \text{ but } e^{rx} \neq 0 \text{ for all } x, \text{ so } \underline{ar^2 + br + c = 0}$$

• This is called the auxiliary eq. There will be three forms of the general solution.

• Case (1): If $\Delta = b^2 - 4ac > 0$, then we have two real and distinct r_1, r_2 | $r_1 \neq r_2$.

$\rightarrow$ So, we have two solutions - $y_1 = e^{r_1 x}$, $y_2 = e^{r_2 x}$
general solution $\rightarrow y = C_1 y_1 + C_2 y_2$

• Case (2): If $\Delta = 0$, then we have two solution with $r_1 = r_2$

→ So, we have two solutions: $y_1 = e^{r_1 x}$, $y_2 = x e^{r_1 x}$

Pf:- $\Delta = 0 \Rightarrow r = \frac{-b \pm 0}{2a} = \frac{-b}{2a}$

$$\Rightarrow r_1 = \frac{-b}{2a} \Rightarrow 2r_1 = \frac{-b}{a}$$

and $ay'' + by' + cy = 0 \Rightarrow y'' + \frac{b}{a}y' + \frac{c}{a}y = 0$
By reduction of order.

$$y_2 = y_1 \int \frac{e^{-\int p(x) dx}}{(y_1)^2} dx = e^{r_1 x} \int \frac{e^{\int \frac{-b}{a} dx}}{(y_1)^2} dx$$

$$y_2 = e^{r_1 x} \int \frac{e^{2r_1 x}}{e^{2r_1 x}} dx = e^{r_1 x} \int \frac{e^{2r_1 x}}{e^{2r_1 x}} dx$$

$$y_2 = e^{r_1 x} \int 1 dx \Rightarrow \boxed{x \cdot e^{r_1 x}}$$

$$ay'' + by' + cy = 0$$

$\downarrow \quad \downarrow \quad \downarrow$
 $r^2 \quad r \quad 1$

-3 (if $r_1 \neq r_2$)

$y \rightarrow 1$
 $y' \rightarrow r$
 $y'' \rightarrow r^2$
 $y''' \rightarrow r^3$
 $y^{(4)} \rightarrow r^4$
...

Ex: solve $y'' + 5y' + 6y = 0$, with I.v.p $y(0) = 1$, $y'(0) = 1$
- find $y(1)$?

Sol: $r^2 + 5r + 6 = 0$

$$(r+3)(r+2) = 0 \rightarrow r_1 = -3, r_2 = -2$$

$$y_1 = e^{-3x}, y_2 = e^{-2x}$$

→ general solution (g.s) $y = C_1 e^{-3x} + C_2 e^{-2x}$

Ex: Solve: $y'' - 8y' + 15y = 0$

Sol: $r^2 - 8r + 15 = 0$, $\Delta = 64 - 4(15)(1) = 4 > 0$

$$(r-3)(r-5) = 0 \rightarrow r_1 = 3, r_2 = 5$$

$$y_1 = e^{3x}, y_2 = e^{5x}$$

→ g.s $y = C_1 e^{3x} + C_2 e^{5x}$

$$y(0) = 1 \Rightarrow 1 = C_1 e^0 + C_2 e^0 \Rightarrow \boxed{1 = C_1 + C_2} \rightarrow \textcircled{1}$$

$$y'(0) = -1 \Rightarrow y' = 3e^{3x}C_1 + 5e^{5x}C_2 \Rightarrow -1 = 3C_1 e^0 + 5C_2 e^0$$

$$\boxed{-1 = 3C_1 + 5C_2} \rightarrow \textcircled{2}$$

$$\boxed{y = 3e^{3x} - 2e^{5x}}$$

$$-3 = -3C_1 + -3C_2$$

$$-1 = 3C_1 + 5C_2$$

$$-4 = 2C_2$$

$$\boxed{C_2 = -2}$$

$$\boxed{C_1 = 3}$$

Ex 3 $y'' - 4y' + 4y = 0$

Sol 3 $r^2 - 4r + 4 = 0$, $\Delta = 0$

$$(r-2)(r-2) = 0$$

$$r = 2, 2$$

$$y_1 = e^{2x}, y_2 = x \cdot e^{2x}$$

(g.s) $\rightarrow y = C_1 e^{2x} + C_2 x \cdot e^{2x}$

Ex 3 solve $y''' - 5y'' + 6y' = 0$

Sol 3 $r^3 - 5r^2 + 6r = 0 \rightarrow r(r^2 - 5r + 6) = 0$

$$r(r-2)(r-3) = 0$$

$$r_1 = 0, r_2 = 2, r_3 = 3$$

$$y_1 = e^{0x}, y_2 = e^{2x}, y_3 = e^{3x}$$

(g.s) $\rightarrow y = C_1 + C_2 e^{2x} + C_3 e^{3x}$

Ex 3 - If $y = C_1 + C_2 e^{3x}$ is the general solution for the second order D.E find D.E ?!

Sol 3 $r_1 = 0, r_2 = 3 \rightarrow r(r-3) = 0$

$$r^2 - 3r = 0$$

$$\boxed{y'' - 3y' = 0}$$

Ex 3- If $y = X \cdot e^{4x}$ is a solution for second D.E find the D.E?

Sol 3- $r = 4, 4 \rightarrow (r-4)(r-4) = 0$
 $r^2 - 8r + 16 = 0 \rightarrow y'' - 8y' + 16y = 0$

Ex 3- solve $y''' - 8y'' + 16y' - 8y = 0$

Sol 3- $r^3 - 8r^2 + 16r - 8 = 0$

* بتجربة نعرف رقم ليس
 مانقله الى اليمين

$2^3 - 8 \cdot 2^2 + 16(2) - 8 = 0 \checkmark$

$(r-2)(r^2 - 6r + 4) = 0$

$r = 2, r = \frac{6 \pm 4\sqrt{2}}{2}$

$r = 3 + 2\sqrt{2}$

$r = 3 - 2\sqrt{2}$

2	1	-8	16	-8
	0	2	-12	8
	1	-6	4	0

$r^2 - 6r + 4 = 0$

$r = \frac{6 \pm \sqrt{36-4}}{2}$

$r = \frac{6 \pm \sqrt{32}}{2}$

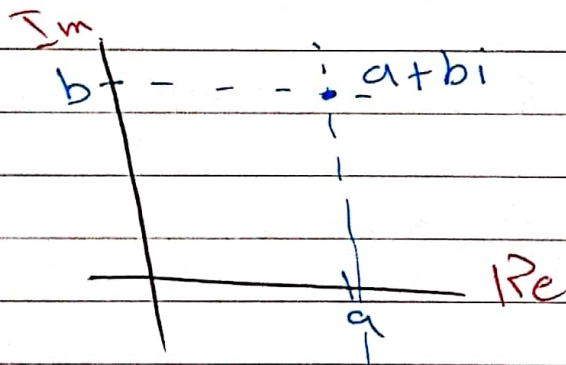
• Case (3) If $\Delta < 0$, then we have complex roots $\alpha \pm \beta i$, so we have two solutions:-

$$y_1 = e^{\alpha x} \cos \beta x, \quad y_2 = e^{\alpha x} \sin \beta x$$

→ The general solution:- $y = C_1 y_1 + C_2 y_2$.

Complex number:- A complex number consist of a real part and an imaginary part.

$$\text{real part } \underbrace{a}_{\text{real part}} + \underbrace{bi}_{\text{imaginary part}}$$



Ex:- 1) $\sqrt{-4} = 2i$ 2) $\sqrt{-49} = 7i$ 3) $\sqrt{-10} = \sqrt{10}i$

Ex:- 1) solve $x^2 + 2x + 3 = 0$

$$\Delta = 4 - 4(3) = -8 \Rightarrow x = \frac{-2 \pm \sqrt{-8}}{2} = \frac{-2 \pm \sqrt{8}i}{2}$$

$$x = \frac{-2}{2} \pm \frac{\sqrt{8}}{2}i \Rightarrow \boxed{x = -1 \pm \sqrt{2}i}$$

2) solve $x^2 + 3x + 4 = 0$ $\Delta = 9 - 16 = -7$

$$x = \frac{-3 \pm \sqrt{-7}}{2} \Rightarrow x = \frac{-3}{2} \pm \frac{\sqrt{7}}{2}i$$

Ex 3 - solve the D.E $y''' + 3y'' + 4y' = 0$

Sol 3 - $r^3 + 3r^2 + 4r = 0 \rightarrow r(r^2 + 3r + 4) = 0$

$$r_1 = 0, \Delta = 9 - 16 = -7$$

$$r = \frac{-3 \pm \sqrt{-7}}{2}$$

$$y_1 = e^{0x} = 1$$

$$y_2 = e^{-\frac{3}{2}x} \cos \frac{\sqrt{7}}{2}x$$

$$y_3 = e^{-\frac{3}{2}x} \sin \frac{\sqrt{7}}{2}x$$

$$\leftarrow r = \frac{-3 \pm \sqrt{7}}{2} ;$$

Ex 3 - solve the D.E $y'' + y' + 3y = 0$

Sol 3 - $r^2 + r + 3 = 0 \rightarrow \Delta = 1 - 4(3) = -11 < 0$

$$r = \frac{-1 \pm \sqrt{-11}}{2} \Rightarrow r = \frac{-1 \pm \frac{\sqrt{11}}{2}i}{2}$$

$$y_1 = e^{-\frac{1}{2}x} \cos \frac{\sqrt{11}}{2}x$$

$$y_2 = e^{-\frac{1}{2}x} \sin \frac{\sqrt{11}}{2}x$$

So, g.s $y = C_1 e^{-\frac{1}{2}x} \cos \frac{\sqrt{11}}{2}x + C_2 e^{-\frac{1}{2}x} \sin \frac{\sqrt{11}}{2}x$

Ex 3 solve $y'''(x) + 3y''(x) - 4y(x) = 0$

Sol $r^3 + 3r^2 - 4 = 0 \rightarrow 1^3 + 3 - 4 = 0 \checkmark \underline{r=1}$

$$(r-1)(r^2+4r+4)=0$$

$$(r-1)(r+2)(r+2)=0$$

$$r=1, r=-2, -2$$

1	1	3	0	-4
	0	1	4	4
	1	4	4	0

$$y_1 = e^x, y_2 = e^{-2x}, y_3 = x \cdot e^{-2x}$$

$$\text{g.s.: } y = C_1 e^x + C_2 e^{-2x} + C_3 x e^{-2x}$$

$$\text{Ex 3: solve D.E } y^{(4)} + 2y'' + y = 0$$

$$\text{Sol: } r^4 + 2r^2 + 1 = 0 \Rightarrow (r^2)^2 + 2r^2 + 1 = 0$$

$$(r^2+1)(r^2+1) = 0$$

$$r^2+1=0 \rightarrow r^2=-1 \rightarrow r=\pm\sqrt{-1} \Rightarrow r=0 \pm i, \alpha=0, \beta=1$$

$$r^2+1=0 \rightarrow r=\pm\sqrt{-1} \Rightarrow r=0 \pm i, \alpha=0, \beta=1$$

$$\rightarrow y_1 = e^{0x} \cos x = \cos x, y_2 = e^{0x} \sin x = \sin x$$

$$y_3 = x e^{0x} \cos x = x \cos x, y_4 = x e^{0x} \sin x = x \sin x$$

$$\text{g.s.: } y = C_1 \cos x + C_2 \sin x + C_3 x \cos x + C_4 x \sin x$$

$$\text{Ex 3: solve } y^{(4)} + 4y'' + y' - 16y' - 12y = 0$$

$$\text{Sol: } r^4 + 4r^3 - r^2 - 16r - 12 = 0$$

$$(-1)^4 + 4(-1)^3 - (-1)^2 - 16(-1) - 12 \stackrel{?}{=} 0 \rightarrow 12 - 12 = 0 \checkmark \boxed{r=-1}$$

$$(r+1) = 0 \Rightarrow \boxed{r = -1}$$

1	4	-1	-16	-12
0	-1	-3	4	12
1	3	-4	-12	0

$$(r+1)(r^3+3r^2-4r-12)=0 \rightarrow (2)^3+2(2)^2-4(2)-12 \stackrel{?}{=} 0 \checkmark$$

$$(r+1)(r-2) \quad \boxed{r=2}$$

1	3	-4	-12
0	2	10	12
1	5	6	0

$$(r+1)(r-2)(r^2+5r+6)=0$$

$$(r+1)(r-2)(r+2)(r+3)=0$$

$$r = -1, 2, -2, -3$$

$$y_1 = e^{-x}, y_2 = e^{2x}, y_3 = e^{-2x}, y_4 = e^{-3x}$$

$$\text{g.s.: } y = C_1 e^{-x} + C_2 e^{2x} + C_3 e^{-2x} + C_4 e^{-3x}.$$

$$\underline{\text{Ex.}} \Rightarrow \text{solve } y^{(5)}(x) = 0$$

$$\underline{\text{sol.}} \quad r^5 = 0 \rightarrow r = 0, 0, 0, 0, 0$$

$$(r-0)(r-0)(r-0)(r-0)(r-0) = 0$$

$$y_1 = e^{0x} = 1, y_2 = x e^{0x} = x, y_3 = x^2 e^{0x} = x^2, y_4 = x^3 e^{0x} = x^3$$

$$y_5 = x^4 e^{0x} = x^4$$

$$\text{g.s.: } y = C_1 + C_2 x + C_3 x^2 + C_4 x^3 + C_5 x^4.$$

Ex 3- Given $(r+2)(r-4)^2(r+25)^2(r^2-5r+6)=0$ is the auxiliary eq. for Homog. D.E with constant coefficient, find the general solution?

Sol - $(r+2)=0 \rightarrow r=-2, y_1 = e^{-2x}$

$$(r^2-9)^2=0 \rightarrow (r^2-9)(r^2-9)=0$$

$$r=\pm 3, r=\pm 3$$

$$y_2 = e^{3x}, y_3 = e^{-3x}, y_4 = X \cdot e^{3x}, y_5 = X \cdot e^{-3x}$$

$$(r^2+25)^2 = (r^2+25)(r^2+25)=0$$

$$r^2+25=0 \rightarrow r^2 = -25 \rightarrow r = \pm \sqrt{-25} \rightarrow r = 0 \pm 5i$$

$$\alpha=0, \beta=5$$

$$y_6 = e^{0x} \cos 5x = \cos 5x, y_7 = e^{0x} \sin 5x = \sin 5x$$

$$r^2+25=0 \rightarrow r = 0 \pm 5i \quad \alpha=0, \beta=5$$

$$y_8 = X \cos 5x, y_9 = X \sin 5x$$

$$(r^2-5r+6)=0 \Rightarrow (r-2)(r-3)=0 \quad \underline{r=2,3}$$

$$y_{10} = e^{2x}, y_{11} = X^2 \cdot e^{3x}$$

$$\text{g.s. } y = C_1 e^{-2x} + C_2 e^{3x} + C_3 e^{-3x} + C_4 X e^{3x} + C_5 X e^{-3x} \\ + C_6 \cos 5x + C_7 \sin 5x + C_8 X \cos 5x + C_9 X \sin 5x \\ + C_{10} e^{2x} + C_{11} X^2 e^{3x}.$$

Ex 3- If $y = C_1 X + C_2 X \cos 4X + C_3 e^{4X}$ is a (eg. s) for the Homo. D.E, find the D.E?

Sol 3- $r=0,0$, $r=0 \pm 4i$, $r=0 \pm 4i$

$$e^{0X} \cos 4X = \cos 4X \quad \left| \begin{array}{l} r=0 \pm 4i \\ r^2 = (4i)^2 \rightarrow r^2 = -16 \\ r^2 + 16 = 0 \end{array} \right. \quad \begin{array}{l} (i)^2 = -1 \\ i \cdot i = -1 \end{array}$$

$$(r-0)(r-0)(r-4i)(r^2+16)(r^2+16)=0$$

$$r^2(r-4i)(r^2+16)(r^2+16)=0$$

$$(r^3 + 4ir^2)(r^4 + 32r^2 + 256)=0$$

$$r^7 + 32r^5 + 265r^3 - 4r^6 - 128r^4 - 1624r^2 = 0$$

$$y^{(7)} - 4y^{(6)} + 32y^{(5)} - 128y^{(4)} + 256y''' - 1624y'' = 0$$

Ex 3- If $y = e^{2X} \sin 3X$ is a solution for $ay'' + by' + cy = 0$, find a, b, c ?

Sol 3- $r=2 \pm 3i$

$r=2 \mp 3i$

$$(r-2) = \pm 3i \rightarrow (r-2)^2 = (\pm 3i)^2 \rightarrow r^2 - 4r + 4 = -9$$

$$r^2 - 4r + 13 = 0 \rightarrow y'' - 4y' + 13y = 0$$

So, $a=1$, $b=-4$, $c=13$

■ 1 hr m 3. let y_1 be the solution for D.E
 $a(x)y'' + b(x)y' + c(x)y = f(x)$, and
 y_2 be a solution of O.D.E
 $a(x)y'' + b(x)y' + c(x)y = g(x)$, Then the
 solution of $a(x)y'' + b(x)y' + c(x)y = C_1 f(x) + C_2 g(x)$
 is $\rightarrow \underline{C_1 y_1 + C_2 y_2}$.

Ex³- let y_1 be a solution for $y'' + x^2 y' + y = x^2 - 2x$
 and y_2 be a solution $y'' + x^2 y' + y = 2x^2 + 11x + 9$
 $5x + 3$

find the solution for $y'' + x^2 y' + y = 2x^2 + 11x + 9$.

Sol³- $C_1 f(x) + C_2 g(x) = 2x^2 + 11x + 9$

$$C_1(x^2 - 2x) + C_2(5x + 3) = 2x^2 + 11x + 9$$

$$C_1 x^2 - C_1 2x + C_2 5x + 3C_2 = 2x^2 + 11x + 9$$

comparing L.H.S

$$x^2: \underline{C_1 = 2}$$

$$x: -2C_1 + 5C_2 = 1$$

$$5C_2 = 15 \Rightarrow \underline{C_2 = 3}$$

So, the general solution is:-

So, the solution is:-

$$\underline{2y_1 + 3y_2}$$

* Euler-equation

Def³ - Cauchy-Euler eq is ODE of the form -

$$a_n x^n \cdot y^{(n)} + a_{n-1} x^{n-1} \cdot y^{(n-1)} + \dots + a_1 x y' + a_0 y$$

where $a_0, a_1, \dots, a_n$ are constant is said to be a Cauchy-Euler-equation

* Cauchy second-order $[n=2]$ $ax^2y'' + bxy' + cy = 0$

Ex³ - 1] $x^2y'' + xy' + cy = 0$ Cauchy.

2] $x^3y'' + xy' = 0$ Cauchy.

3] $x^4y^{(5)} + x^2y'' + y' = 0$ non-cauchy.

4] $xy'' + y' = 0, x > 0$ Cauchy

* Now, To solve the Cauchy-Euler eq convert to D.E with constant-coefficient by substitution:

$$[t = \ln(x)] , [x = e^t]$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \left(\frac{dt}{dx} \right) = \frac{dy}{dt} \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{1}{x}$$

$$\frac{d^2y}{dx^2} = \frac{dy}{dt} \left(-\frac{1}{x^2} \right) + \frac{1}{x} \left(\frac{d^2y}{dt^2} \right) \frac{1}{x}$$

$$\frac{d^2y}{dx^2} = \frac{dy}{dt} \left(-\frac{1}{x^2} \right) + \frac{1}{x^2} \left(\frac{d^2y}{dt^2} \right)$$

$$a x^2 y'' + b x y' + c y = 0$$

$$a \cancel{x^2} \left(\frac{1}{\cancel{x^2}} \left(-\frac{dy}{dt} + \frac{d^2y}{dt^2} \right) \right) + b \cancel{x} \left(\frac{dy}{dt} \cdot \frac{1}{\cancel{x}} \right) + c y = 0$$

$$-a \frac{dy}{dt} + a \frac{d^2y}{dt^2} + b \frac{dy}{dt} + c y = 0$$

$$a \frac{d^2y}{dt^2} + (b-a) \frac{dy}{dt} + c y = 0$$

$$| a y'' + (b-a) y' + c y = 0 |$$

Ex 3: Solve $x^2 y'' - 2xy' - 4y = 0$?!

Sol: $t = \ln(x)$, $a=1$, $b=-2$, $c=-4$

$$ay'' + (b-a)y' + cy = 0 \rightarrow 1 \cdot y'' + (-2-1)y' - 4y = 0$$

$$y'' - 3y' - 4y = 0 \rightarrow r^2 - 3r - 4 = 0 \quad y_1 = e^{4t}$$

$$(r-4)(r+1) = 0$$

$$\underline{r=4, -1}$$

$$y_2 = e^{-t}$$

$$y_1 = e^{4 \ln(x)}$$

$$\text{g.s} \rightarrow y = C_1 x^4 + C_2 x^{-1}$$

$$y_1 = x^4, y_2 = x^{-1}$$

Ex 3: $4x^2 y'' + 8xy' + y = 0$

Sol: $t = \ln(x)$, $a=4$, $b=8$, $c=1$

$$4y'' + 4y' + y = 0 \rightarrow 4r^2 + 4r + 1 = 0$$

$$(2r+1)(2r+1) = 0$$

$$r = -\frac{1}{2}, -\frac{1}{2}$$

$$y_1 = e^{-\frac{1}{2}t} = e^{-\frac{1}{2} \ln(x)} = x^{-\frac{1}{2}}$$

$$y_2 = t \cdot e^{-\frac{1}{2}t} = \ln(x) \cdot e^{-\frac{1}{2} \ln(x)} = \ln(x) \cdot x^{-\frac{1}{2}}$$

$$\text{g.s} \rightarrow y = C_1 x^{-\frac{1}{2}} + C_2 \ln(x) \cdot x^{-\frac{1}{2}}$$

Ex 3 - $x^2 y'' - xy' + 2y = 0$

Sol 3 - $t = \ln(x), a=1, b=-1, c=2$

$$y'' - 2y' + 2y = 0 \rightarrow r^2 - 2r + 2 = 0$$

$$D = 4 - 4(2) = -4 < 0 \rightarrow r = \frac{2 \pm \sqrt{-4}}{2}$$

$$r = 1 \pm i \quad | \alpha = 1, \beta = 1 |$$

$$y_1 = e^t \cdot \cos t, y_2 = e^t \sin t$$

$$y_1 = e^{\ln(x)} \cdot \cos \ln(x) \rightarrow y_1 = x \cdot \cos \ln(x)$$

$$y_2 = e^{\ln(x)} \sin \ln(x) \rightarrow y_2 = x \cdot \sin \ln(x)$$

g.s $\rightarrow y = C_1 x \cos \ln(x) + C_2 x \sin \ln(x)$

Ex 3 - Determine the Cauchy-Euler having a g.c $y = x^2 \cos 3 \ln(x)$?

Sol 3 - $\alpha = 2, \beta = 3 \quad r = 2 \pm 3i \rightarrow (r-2)^2 = -9$

$$r^2 - 4r + 4 = -9 \rightarrow r^2 - 4r + 13 = 0$$

$$y'' - 4y' + 13y = 0$$

$$ax^2 y'' + bxy' + cy = 0$$

$$x^2 y'' - 3xy' + 13y = 0$$

$a = 1$ $b - a = -4$ $b = -3$ $c = 13$

Ex 3- Determine Cauchy-Euler eq of second order
homo. having g.s.s- $y = C_1 x + C_2 \ln(x)$?

sol 3- $r = 0, 0 \rightarrow (r-0)(r-0) = 0$

$$r^2 = 0 \rightarrow y'' = 0$$

$$a=1, b-a=0, c=0, b=1$$

$$x^2 y'' + x y' = 0$$

* Undetermined coefficient:-

solving non-homo ODE, $\underline{g(x) \neq 0}$

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = g(x)$$

which can be written as operator:-

$$L[y](x) = g(x) \quad | \quad L[y](x) = y'' + p(x)y' + q(x)y$$

So, $L[y](x) = y'' + p(x)y' + q(x)y$, if we view L as an operator that transform a twice DE. $y = y(x)$ into C-S function for any two twice D² function $y_1(x), y_2(x)$

$$L[y_1(x) + y_2(x)] = (y_1 + y_2)'' + p(x)[y_1 + y_2]' + q(x)[y_1 + y_2]$$

$$= y_1'' + y_2'' + p(x)y_1' + p(x)y_2' + q(x)y_1 + q(x)y_2$$

$$= y_1'' + p(x)y_1' + q(x)y_1 + y_2'' + p(x)y_2' + q(x)y_2$$

$$= L[y_1(x)] + L[y_2(x)]$$

Ex 3:- $y'' - 2y' + y = e^x$

Sol:- $L[y](x) = e^x$ non-homo.

$$L[y](x) = 0 \quad \text{homo.}$$

• To solve the non-homo ODE we have two methods:-

1] we solve Homo ODE $[y](x)=0$ say complementary solution y_c .

2] we solve the non-homo $[y]=g(x)$ say the particular solution y_p .

* The general solution $\rightarrow \underline{y = y_c + y_p}$

Ex 3. solve $y'' - 5y' + 6y = 12x + 14$

sol:- $y'' - 5y' + 6y = 0 \rightarrow r^2 - 5r + 6 = 0 \rightarrow (r-2)(r-3) = 0$
 $y_1 = e^{2x}, y_2 = e^{3x} \Rightarrow y_c = C_1 e^{2x} + C_2 e^{3x}$

now, we want to find $y_p(x)$:- $y_p(x) = Ax + B$

$$y'_p(x) = A \text{ and } y''_p(x) = 0$$
$$y'' - 5y' + 6y = 12x + 14$$

$$0 - 5A + 6(Ax + B) = 12x + 14 \rightarrow -5A + \underbrace{6Ax} + \underbrace{6B} = 12x + 14$$
$$6Ax = 12x \quad \leftarrow \text{coefficient}$$
$$6A = 12 \rightarrow \underline{A = 2}$$

$$-5A + 6B = 14$$

$$-5(2) + 6B = 14 \rightarrow \underline{B = 4} \rightarrow y_p(x) = 2x + 4$$

$$\text{general solution} \rightarrow y = C_1 e^{2x} + C_2 e^{3x} + 2x + 4.$$

Ex: solve $y'' - y = 2x - 4$ non-homo

sol:- $y'' - y = 0 \rightarrow r^2 - 1 = 0 \Rightarrow r = \pm 1$

$$y_1 = e^x, y_2 = e^{-x}$$

$$y_c = C_1 e^x + C_2 e^{-x}$$

now, we want to find $y_p(x) = Ax + B$

$$y'_p(x) = A, y''_p(x) = 0$$

$$y'' - y = 2x - 4$$

$$0 - (Ax + B) = 2x - 4 \rightarrow -Ax - B = 2x - 4$$

$$-A = 2$$

مقارنة مع $2x - 4$

$$\underline{A = -2}$$

$$\text{أيضا: } -B = -4 \rightarrow \underline{B = 4} \rightarrow y_p(x) = -2x + 4$$

general solution: $y = C_1 e^x + C_2 e^{-x} - 2x + 4$.

ملاحظة مهمة:- في المثال السابق كان الافتراض $g(x)$ خطي (الخطي)

لذلك نأخذ $y_p(x) = Ax + B$ ولكن إذا كان تربيعي يكون
أو $y_p(x) = Ax^2 + Bx + C$ لذلك الآن نأخذ جدول
توضيحي للاعتراضات $g(x)$

Table 3- $g(x)$ $y_p(x)$

Constant $\longrightarrow A$

polynomial

$3x+1 \longrightarrow Ax+B$

$6x^2+x+2 \longrightarrow Ax^2+Bx+C$

e^{ax}

$3e^{5x} \longrightarrow Ae^{5x}$

$xe^{4x} \longrightarrow (Ax+B)e^{4x}$

$x^2e^{3x} \longrightarrow (Ax^2+Bx+C)e^{3x}$

$\sin ax, \cos ax$

$\frac{1}{2}\sin 4x \longrightarrow A\sin 4x + B\cos 4x$

$x\cos 7x \longrightarrow (Ax+B)\cos 7x + (Cx+D)\sin 7x$

$5e^{2x}\sin 4x \longrightarrow Ae^{2x}\sin 4x + Be^{2x}\cos 4x$

$x^2\sin 2x \longrightarrow (Ax^2+Bx+C)\sin 2x + (Dx^2+Ex+F)\cos 2x$

$xe^{3x}\cos 4x \longrightarrow (Ax+B)e^{3x}\cos 4x + (Cx+D)e^{3x}\sin 4x$

Ex solve $y'' - y = 2 \sin 4x$

sol:- $y'' - y = 0 \rightarrow r^2 - 1 = 0 \rightarrow y_1 = e^x, y_2 = e^{-x}$

$$y_c = C_1 e^x + C_2 e^{-x}$$

now, $y_p(x) = A \cos 4x + B \sin 4x$

$$y'_p(x) = -4A \sin 4x + 4B \cos 4x$$

$$y''_p(x) = -16A \cos 4x - 16B \sin 4x$$

$$y'' - y = 2 \sin 4x \quad \text{نك لعلك وجره}$$

$$(-16A \cos 4x - 16B \sin 4x) - (A \cos 4x + B \sin 4x) = 2 \sin 4x$$

$$\underbrace{-16A \cos 4x - 16B \sin 4x} - \underbrace{A \cos 4x - B \sin 4x} = \underbrace{2 \sin 4x}$$

$$\begin{array}{l|l} -16B - B = 2 \rightarrow B = \frac{-2}{17} & -16A - A = 0 \\ \hline & A = 0 \end{array}$$

$$y_p(x) = 0 \cdot \cos 4x - \frac{2}{17} \sin 4x = -\frac{2}{17} \sin 4x$$

Ex 3 - solve $y'' - 2y' + y = e^x$.

$$r^2 - 2r + 1 = 0 \rightarrow (r-1)(r-1) = 0$$

$$r = 1, 1$$

$$y_1 = e^x, y_2 = xe^x \rightarrow y_c = C_1 e^x + C_2 x e^x$$

$$\text{now, } y_p(x) = A x^2 e^x$$

فقرنا ان y_p به x^2 وذلك لان $r=1$ (المرتبة 2)
المرتبة 2

$$y'_p(x) = 2A x e^x + A x^2 e^x$$

$$y''_p(x) = 2A x e^x + 2A e^x + A x^2 e^x + 2A x e^x$$

$$y''_p(x) = 4A x e^x + 2A e^x + A x^2 e^x$$

$$y'' - 2y' + y = e^x \text{ نضعها في المعادلة}$$

$$4A x e^x + 2A e^x + A x^2 e^x - 2(2A x e^x + A x^2 e^x) + A x^2 e^x = e^x$$

$$\Rightarrow \text{بالنسبة لـ } e^x: 2A = 1 \rightarrow \underline{A = \frac{1}{2}} \quad y_p(x) = \frac{1}{2} x^2 e^x$$

$$\text{general solution: } y = C_1 e^x + C_2 x e^x + \frac{1}{2} x^2 e^x$$

Ex 3:- $y^{(4)} - y'' = x + e^x$

Sol:- $y^{(4)} - y'' = 0 \rightarrow r^4 - r^2 = 0 \rightarrow r^2(r^2 - 1) = 0$
 $r = 0, 0, 0, 1$

$y_1 = e^{0x} = 1$

$y_2 = x, y_3 = x^2, y_4 = e^x$

$y_c = C_1 + C_2 x + C_3 x^2 + C_4 e^x$

$y_p(x) = (Ax + B)x^3 + Cx e^x = Ax^4 + Bx^3 + Cx e^x$

Ex 3:- $y'' + 4y = \cos^2 x$

Sol:- $y'' + 4y = \frac{1}{2} + \frac{1}{2} \cos 2x$

$r^2 + 4 = 0 \rightarrow r^2 = -4 \rightarrow r = \pm \sqrt{-4} = \pm 2i$

$y_1 = e^{0x} \cos 2x = \cos 2x, y_2 = \sin 2x$

$y_c = C_1 \cos 2x + C_2 \sin 2x$

$y_p(x) = A + Bx \sin 2x + Cx \cos 2x$

Ex 3:- $y''' + 4y'' + 3y' = x^2 \cos x - 3x$

Sol:- $r^3 + 4r^2 + 3r = 0 \rightarrow r(r^2 + 4r + 3) = 0$
 $r(r+3)(r+1) = 0$
 $r = 0, -3, -1$

$$y_1 = e^{0x} = 1, y_2 = e^{-x}, y_3 = e^{-3x}$$

$$y_c = C_1 + C_2 e^{-x} + C_3 e^{-3x}$$

$$y_p(x) = (A_1 x^2 + B_1 x + C_1) \cos x + (A_2 x^2 + B_2 x + C_2) \sin x + (C_3 x + B_3) \cdot x$$

$$\text{Ex}^3 - y'' + y = \sin x \cos 2x$$

$$\text{Sol}^3 - y'' + y = \frac{1}{2} [\sin(-x) + \sin(2x)]$$

$$y'' + y = -\frac{1}{2} \sin x + \frac{1}{2} \sin 2x$$

$$y'' + y \rightarrow r^2 + 1 = 0 \rightarrow r = \pm i$$

$$y_1 = \cos x, y_2 = \sin x$$

$$y_c = C_1 \cos x + C_2 \sin x$$

$$y_p(x) = A x \sin x + B x \cos x + C \sin 3x + D \cos 3x$$

$$\text{Ex}^3 - \text{solve } x^2 y'' - x y' + y = 2x$$

$$\text{Sol}^3 - a=1, b=-1, C=1 \quad t = \ln x, x = e^t$$

$$y'' - 2y' + y = 2e^t$$

$$y'' - 2y' + y = 0 \rightarrow r^2 - 2r + 1 = 0 \rightarrow (r-1)(r-1) = 0$$

$$y_1 = e^t, y_2 = t e^t$$

$$r = 1, 1$$

$$y_c = C_1 e^t + C_2 t e^t$$

$$y_p(x) = A t^2 e^t \rightarrow y_p(x) = A x (\ln x)^2 \rightarrow \text{form.}$$

$$\text{g.s} \rightarrow y = C_1 x + C_2 x \ln x + x (\ln x)^2$$

$$y'_p = A t^2 e^t + 2 A t e^t$$

$$y''_p = A t^2 e^t + 2 A t e^t + 2 A t e^t + 2 A e^t$$

$$A t^2 e^t + 4 A t e^t + 2 A e^t - 2(A t^2 e^t + 2 A t e^t) + A t^2 e^t = 2 e^t$$

$$2A = 2 \rightarrow \boxed{A=1}$$

$$y_p(x) = x (\ln x)^2$$

$$\text{Ex 3- Solve } x^2 y'' - x y' + y = \ln x$$

$$\text{Sol} \rightarrow t = \ln x, \quad y'' - 2y' + y = t$$

$$r^2 - 2r + 1 = 0 \rightarrow (r-1)(r-1) = 0$$

$$r = 1, 1$$

$$y_c = C_1 e^t + C_2 t e^t$$

$$y_p = A t + B \rightarrow \boxed{A \ln x + B}$$

$$y'_p = A, \quad y''_p = 0 \quad \text{ن. ل. ل. (جواب)} \rightarrow 0 - 2A + A t + B = t$$

$$\boxed{A=1}$$

$$-2A + B = 0$$

$$\boxed{B=2}$$

$$\text{g.s} \rightarrow y = C_1 x + C_2 x \ln x + \ln x + 2$$

$$y_p = \ln x + 2$$

Ex:- Find the D.E has a general sol
 $y = C_1 x + C_2 x \ln x + \ln x + 2$?

Sol:- $r=1,1 \rightarrow (r-1)(r-1)=0 \rightarrow r^2-2r+1=0$

$$y''-2y'+y=0 \rightarrow a=1, b-a=-2, b=-1, c=1$$

$$x^2 y'' - x y' + y = g(x) \rightarrow \ln x + 2$$

$$y_p = \ln x + 2$$

$$y'_p = \frac{1}{x}, y''_p = -\frac{1}{x^2} \quad \text{N.B. \& \# \& \# \& \#}$$

$$x^2 \left(-\frac{1}{x^2} \right) - x \left(\frac{1}{x} \right) + \ln x + 2 = g(x)$$

$$-1 - 1 + \ln x + 2 = g(x)$$

$$\boxed{g(x) = \ln x}$$

So, The D.E is $\rightarrow x^2 y'' - x y' + y = \ln x$

* Method of variation of parameter :-

If we have second-order non-homo $a(x)y'' + p(x)y' + q(x)y = f(x)$
and $a(x)=1$, for any function $f(x)$.

To solve the D.E by method, we follow:-

① solve the Homo to find y_1 and y_2

$$② y_p(x) = V_1 y_1 + V_2 y_2$$

To find V_1 and V_2 Evaluate:-

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

$$V_1 = \int \frac{W_1}{W} \cdot dx, \quad V_2 = \int \frac{W_2}{W} \cdot dx$$

Ex-3 - solve $y'' + y = \tan x$

Sol:- $r^2 + 1 = 0 \rightarrow r^2 = -1 \rightarrow r = \pm i$

$$y_1 = \cos x, \quad y_2 = \sin x$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$W_1 = \begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix} = 0 - \sin x \tan x$$

$$W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix} = \cos x \tan x - 0 = \sin x$$

$$V_1 = \int \frac{-\sin x \tan x}{1} \cdot dx \rightarrow \int \frac{-\sin^2 x}{\cos x} \cdot dx \rightarrow \int \frac{\cos^2 x - 1}{\cos x} \cdot dx$$

$$V_1 = \int \cos x - \sec x \cdot dx = \sin x - \ln |\sec x + \tan x|$$

$$V_2 = \int \frac{\sin x}{1} \cdot dx = -\cos x$$

$$y_p = V_1 y_1 + V_2 y_2 \rightarrow y_p = (\sin x - \ln |\sec x + \tan x|) \cos x - \cos x \sin x$$

Five Apple

Ex-3 solve $y'' - 2y' + y = \frac{e^x}{x^2 + 1}$

Sol: $r^2 - 2r + 1 = 0 \rightarrow (r-1)(r-1) = 0$

$y_1 = e^x, y_2 = xe^x$

$w = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = \cancel{xe^{2x}} + e^{2x} - \cancel{xe^{2x}} = e^{2x}$

$w_1 = \begin{vmatrix} 0 & xe^x \\ \frac{e^x}{x^2+1} & e^x + e^x \end{vmatrix} = 0 - \frac{xe^{2x}}{x^2+1}$

$w_2 = \begin{vmatrix} e^x & 0 \\ e^x & \frac{e^x}{x^2+1} \end{vmatrix} = \frac{e^{2x}}{x^2+1}$

$V_1 = \int \frac{-xe^{2x}}{x^2+1} \cdot \frac{1}{e^{2x}} dx = \int \frac{-x}{x^2+1} = -\frac{1}{2} \ln(x^2+1)$

$V_2 = \int \frac{\frac{e^{2x}}{x^2+1}}{e^{2x}} dx = \int \frac{1}{x^2+1} dx = \tan^{-1}(x)$

$y_p(x) = V_1 y_1 + V_2 y_2$

$y_p(x) = -\frac{1}{2} \ln(x^2+1) \cdot e^x + xe^x \tan^{-1} x$

* Laplace Transform.

Def 3- Let f be a function defined for $t \geq 0$.
Then, The Integral:-

$$L(f(t)) = \int_0^{\infty} e^{-st} \cdot f(t) \cdot dt.$$

is said to be the Laplace transform of f ,
provided that the integral converges.

$$\rightarrow L(f(t)) = F(s).$$

$$\text{Ex 3- } \textcircled{1} L(1) = \int_0^{\infty} e^{-st} \cdot (1) \cdot dt = \lim_{b \rightarrow \infty} \int_0^b e^{-st} \cdot dt$$

$$\lim_{b \rightarrow \infty} \left. \frac{e^{-st}}{s} \right|_0^b = \lim_{b \rightarrow \infty} \frac{e^{-sb}}{s} + \frac{1}{s} = \frac{e^{-\infty}}{s} + \frac{1}{s}$$

$$= \frac{1}{e^{\infty} \cdot s} + \frac{1}{s} = 0 + \frac{1}{s} = \frac{1}{s}, \quad s > 0, \text{ when}$$

$s > 0$, the exponent $(-sb)$ is negative and $e^{-sb} \rightarrow 0$
as $b \rightarrow \infty$. The Integral diverge for $s < 0$.

* Table

$f(t)$

$F(s) = \mathcal{L}(f(t))$

$$a \longrightarrow \frac{a}{s}$$

$$t^n \xrightarrow{n=1,2,3,\dots} \frac{n!}{s^{n+1}}$$

$$e^{at} \longrightarrow \frac{1}{s-a}$$

$$\sin at \longrightarrow \frac{a}{s^2 + a^2}$$

$$\cos at \longrightarrow \frac{s}{s^2 + a^2}$$

$$\sinh at \longrightarrow \frac{a}{s^2 - a^2}$$

$$\cosh at \longrightarrow \frac{s}{s^2 - a^2}$$

Ex 3- ① $\mathcal{L}(7) = \frac{7}{s}$

② $\mathcal{L}(t) = \frac{1!}{s^{1+1}} = \frac{1}{s^2}$

③ $\mathcal{L}(t^3) = \frac{3!}{s^{3+1}} = \frac{6}{s^4}$ 3! = 3 * 2 * 1

④ $\mathcal{L}(t^{100}) = \frac{100!}{s^{101}}$

⑤ $\int_0^\infty e^{-st} \cdot t^4 \cdot dt = \mathcal{L}(t^4) = \frac{4!}{s^5}$

⑥ $\int_0^\infty e^{-st} \cdot \sin 2\pi t \cdot dt = \mathcal{L}(\sin 2\pi t) = \frac{2\pi}{s^2 + 4\pi^2}$

$$\textcircled{7} \int_0^{\infty} e^{-st+3} e^{4t-2} dt = \int_0^{\infty} e^{-st} \cdot e^3 \cdot e^{4t} \cdot e^{-2} dt = \int_0^{\infty} e^{-st} e^{4t+1} dt$$

$$= \mathcal{L}(e^{4t+1}) = \int_0^{\infty} e^{-st} \cdot e^{4t} \cdot e dt = e \mathcal{L}(e^{4t}) = e \cdot \frac{1}{s-4}$$

$$\textcircled{8} \int_0^{\infty} e^{-6t} \cdot t^3 dt = \mathcal{L}(t^3) \Big|_{s=6} = \frac{3!}{s^4} = \frac{6}{6^4} = \frac{1}{6^3} = \frac{1}{216}$$

Thm 3 - II Linearity properties

Let α and β are constants while $f(t)$ and $g(t)$ are function of t , then:-

$$\mathcal{L}(\alpha f(t) + \beta g(t)) = \alpha \mathcal{L}(f(t)) + \beta \mathcal{L}(g(t))$$

* Laplace يوزع على الجمع والطرح و (إذا في) constant بتبقى.

Ex 3- Find the following Laplace:-

$$\textcircled{1} \mathcal{L}(e^{3t} + \cos 7t) = \mathcal{L}(e^{3t}) + \mathcal{L}(\cos 7t) = \frac{1}{s-3} + \frac{s}{s^2+49}$$

$$\textcircled{2} \mathcal{L}(e^{7t+\pi}) = \mathcal{L}(e^{7t} \cdot e^{\pi}) = e^{\pi} \mathcal{L}(e^{7t}) = e^{\pi} \cdot \frac{1}{s-7} = \frac{e^{\pi}}{s-7}$$

$$\begin{aligned} \textcircled{3} \mathcal{L}(\cos^2 t) &= \mathcal{L}\left(\frac{1}{2} + \frac{1}{2} \cos 2t\right) = \mathcal{L}\left(\frac{1}{2}\right) + \frac{1}{2} \mathcal{L}(\cos 2t) \\ &= \frac{\frac{1}{2}}{s} + \frac{1}{2} \cdot \frac{s}{s^2+4} = \frac{1}{2s} + \frac{1}{2(s^2+4)} \end{aligned}$$

$$\textcircled{4} \mathcal{L}(\sin 2t \cos 2t) = \mathcal{L}\left(\frac{1}{2} \sin 4t\right) = \frac{1}{2} \cdot \frac{4}{s^2+16} = \frac{2}{s^2+16}$$

② Shifting property:-

If the Laplace $\mathcal{L}\{f(t)\} = F(s)$ exists, then:-

$$\mathcal{L}\{e^{at} \cdot f(t)\} = \mathcal{L}\{f(t)\} \Big|_{s \rightarrow s-a} = F(s-a)$$

$\cdot s-a \rightarrow s \downarrow \text{shift}$

Ex:- ① $\mathcal{L}\{e^{3t} \cdot t^4\} = \mathcal{L}\{t^4\} \Big|_{s \rightarrow s-3} \rightarrow \frac{4!}{s^5} = \frac{4!}{(s-3)^5}$

② $\mathcal{L}\{e^{-6t} \cdot \sin 4t\} = \mathcal{L}\{\sin 4t\} \Big|_{s \rightarrow s+6} = \frac{4}{(s+6)^2 + 16}$

③ $\mathcal{L}\{e^{\pi t} \cdot \sin 7t\} = \mathcal{L}\{\sin 7t\} = \frac{7}{s^2 + 49} = \frac{7}{(s-\pi)^2 + 49}$

④ $\mathcal{L}\{e^{-3t+8} \cdot \cos(\pi t)\} = \cos \pi \cdot \cos t - \sin \pi \cdot \sin t = -\cos t$

$$= -\mathcal{L}\{e^{-3t+8} \cdot \cos t\} = -\mathcal{L}\{e^{-3t} \cdot e^8 \cdot \cos t\} = -e^8 \mathcal{L}\{e^{-3t} \cdot \cos t\}$$

$$= \frac{e^8 \cdot s}{s^2 + 1} = \frac{e^8 \cdot (s+3)}{(s+3)^2 + 1}$$

⑤ $\mathcal{L}\{e^{-7t} \cdot \cosh 3t\} = \frac{s}{s^2 - 9} = \frac{(s+7)}{(s+7)^2 - 49}$

[3] Derivative of the Laplace Transform.

1 hr m 3 - Let $F(s) = \mathcal{L}\{f(t)\}$, and assume $f(t)$ is piecewise cont. on $[0, \infty)$ and exponential order, Then:-

$$\mathcal{L}\{t^n f(t)\}(s) = (-1)^n \cdot \frac{d^n}{ds^n} (F(s)).$$

Ex 3 ^① $\mathcal{L}\{t \cdot \sin 7t\} = (-1)^1 \cdot \frac{d}{ds} [\mathcal{L}\{\sin 7t\}]$

Sol $= -\frac{d}{ds} \left(\frac{7}{s^2 + 49} \right) = \frac{7(2) \cdot s}{(s^2 + 49)^2} = \frac{14s}{(s^2 + 49)^2}$

^② $\int_0^{\infty} e^{-st} \cdot t \cdot \sin 7t \cdot dt$

Sol $\mathcal{L}\{t \cdot \sin 7t\} = \frac{14s}{(s^2 + 49)^2}$

^③ $\mathcal{L}\{t \cdot \cos 2t\} = (-1) \frac{d}{ds} (\mathcal{L}\{\cos 2t\})$

$$= -\frac{d}{ds} \left(\frac{s}{s^2 + 4} \right) = -\left(\frac{(s^2 + 4) - s(2s)}{(s^2 + 4)^2} \right)$$

$$= \frac{-s^2 - 4 + 2s^2}{(s^2 + 4)^2}$$

* Inverse Laplace Transform.

Def²- Given a function $F(s)$, if there is a function $f(t)$ is cont in $[0, \infty)$ and satisfies $\mathcal{L}(f(t)) = F(s)$. Then, the inverse Laplace of $F(s)$ is-

$$f(t) = \mathcal{L}^{-1}(F(s))$$

Table³- $f(t)$ $\mathcal{L}(f(t)) = F(s)$

$$a \xrightarrow{\quad} \mathcal{L}^{-1}\left(\frac{a}{s}\right)$$

$$t^n \xrightarrow{\quad} \mathcal{L}^{-1}\left(\frac{n!}{s^{n+1}}\right)$$

$$e^{at} \xrightarrow{\quad} \mathcal{L}^{-1}\left(\frac{1}{s-a}\right)$$

$$\cos at \xrightarrow{\quad} \mathcal{L}^{-1}\left(\frac{s}{s^2+a^2}\right)$$

$$\sin at \xrightarrow{\quad} \mathcal{L}^{-1}\left(\frac{a}{s^2+a^2}\right)$$

$$\cosh at \xrightarrow{\quad} \mathcal{L}^{-1}\left(\frac{s}{s^2-a^2}\right)$$

$$\sinh at \xrightarrow{\quad} \mathcal{L}^{-1}\left(\frac{a}{s^2-a^2}\right)$$

$$\text{Ex-3-} \textcircled{1} \mathcal{L}^{-1}\left(\frac{4}{s}\right) = 4 \quad \textcircled{2} \mathcal{L}^{-1}\left(\frac{100}{s}\right) = 100$$

$$\textcircled{3} \mathcal{L}^{-1}\left(\frac{6}{s^3}\right) = \mathcal{L}^{-1}\left(\frac{3 \cdot 2!}{s^3}\right) = 3 \cdot t^2$$

$$\textcircled{4} \mathcal{L}^{-1}\left(\frac{1}{s^4}\right) = \frac{1}{3!} \mathcal{L}^{-1}\left(\frac{3!}{s^4}\right) = \frac{1}{6} t^3.$$

$$\textcircled{5} \mathcal{L}^{-1}\left(\frac{7}{s-10}\right) = 7 \mathcal{L}^{-1}\left(\frac{1}{s-10}\right) = 7 \cdot e^{10t}.$$

$$\textcircled{6} \mathcal{L}^{-1}\left(\frac{1}{s+7}\right) = e^{-7t} \quad \textcircled{7} \mathcal{L}^{-1}\left(\frac{3}{s^2+9}\right) = \sin 3t.$$

$$\textcircled{8} \mathcal{L}^{-1}\left(\frac{1}{s^2+7}\right) = \frac{1}{\sqrt{7}} \mathcal{L}^{-1}\left(\frac{\sqrt{7}}{s^2+7}\right) = \frac{1}{\sqrt{7}} \cdot \sin \sqrt{7}t.$$

Ex-3-

$$\textcircled{1} \mathcal{L}^{-1}\left(\frac{1}{4s^2}\right) = \frac{1}{4 \cdot 2} \mathcal{L}^{-1}\left(\frac{2}{s^3}\right) = \frac{1}{8} t^2.$$

$$\textcircled{2} \mathcal{L}^{-1}\left(\frac{1}{(s-3)^4}\right) = \frac{1}{3!} \mathcal{L}^{-1}\left(\frac{3!}{(s-3)^4}\right) \\ = \frac{1}{6} \underline{e^{3t}} \cdot t^3$$

$$\textcircled{3} \mathcal{L}^{-1}\left(\frac{1}{3s^2+2}\right) = \mathcal{L}^{-1}\left(\frac{1}{3\left(s^2+\frac{2}{3}\right)}\right) = \frac{1}{3} \mathcal{L}^{-1}\left(\frac{1}{\left(s^2+\frac{2}{3}\right)}\right) \\ = \sqrt{\frac{3}{2}} \cdot \frac{1}{3} \left(\frac{\sqrt{\frac{2}{3}}}{s^2+\frac{2}{3}}\right) = \frac{1}{\sqrt{6}} \cdot \sin \sqrt{\frac{2}{3}} t.$$

$$\textcircled{4} \mathcal{L}^{-1}\left(\frac{2}{(s+1)^2+4}\right) = e^{-t} \cdot \sin 2t.$$

$$\text{Ex 3- ① } \mathcal{L}^{-1}\left(\frac{s-4+4}{(s-4)^2+36}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{s-4}{(s-4)^2+36}\right) + \mathcal{L}^{-1}\left(\frac{4}{(s-4)^2+36}\right)$$

$$= e^{4t} \cos 6t + \frac{4}{6} \left(\frac{6}{(s-4)^2+36} \right)$$

$$= e^{4t} \cos 6t + \frac{2}{3} e^{4t} \sin 6t.$$

$$\text{② } \mathcal{L}^{-1}\left(\frac{1}{s^2-5s+6}\right) \quad \text{هو}$$

$$\text{هو الأول الأول} \quad \mathcal{L}^{-1}\left(\frac{1}{(s-3)(s-2)}\right) =$$

$$\frac{1}{(s-3)(s-2)} = \frac{A}{s-3} + \frac{B}{s-2}$$

$$1 = A(s-2) + B(s-3)$$

$$\rightarrow \boxed{s=2}: \underline{B=-1} \quad | \rightarrow \boxed{s=3}: \underline{A=1}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s-3} - \frac{1}{s-2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s-3}\right) - \mathcal{L}^{-1}\left(\frac{1}{s-2}\right)$$

$$= e^{3t} - e^{2t}.$$

- طريقة إكمال المربع - الحل الثاني

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{1}{s^2 - 5s + \frac{25}{4} - \frac{25}{4} + 6}\right) &= \mathcal{L}^{-1}\left(\frac{1}{(s - \frac{5}{2})^2 - \frac{1}{4}}\right) \\ &= 2 \mathcal{L}^{-1}\left(\frac{\frac{1}{2}}{(s - \frac{5}{2})^2 - \frac{1}{4}}\right) = 2 \cdot e^{\frac{5}{2}t} \cdot \sinh \frac{1}{2}t. \end{aligned}$$

$$\left(\frac{\mathcal{L}\{f(t)\}}{2}\right)^2 = \mathcal{L}\{f(t) \cdot f(t)\}$$

$$\textcircled{3} \mathcal{L}^{-1}\left(\frac{1}{s^2 - 4s + 5}\right) =$$

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{1}{s^2 - 4s + 4 - 4 + 5}\right) &= \mathcal{L}^{-1}\left(\frac{1}{(s - 2)^2 + 1}\right) \\ &= e^{2t} \sin t. \end{aligned}$$

$$\text{Ex 3. } \mathcal{L}^{-1} \left(\frac{3}{(s^2+4)(s^2+9)} \right)$$

$$\text{Sol. } -\frac{3}{5} \mathcal{L}^{-1} \left(\frac{[(s^2+4)-(s^2+9)]}{(s^2+4)(s^2+9)} \right)$$

$$-\frac{3}{5} \mathcal{L}^{-1} \left(\frac{\cancel{s^2+4}}{(\cancel{s^2+4})(s^2+9)} - \frac{\cancel{s^2+9}}{(s^2+4)(\cancel{s^2+9})} \right)$$

$$-\frac{3}{5} \mathcal{L}^{-1} \left(\frac{1}{s^2+9} - \frac{1}{s^2+4} \right)$$

$$-\frac{3}{5} \left[\frac{1}{3} \mathcal{L}^{-1} \left(\frac{3}{s^2+9} \right) - \frac{1}{2} \mathcal{L}^{-1} \left(\frac{2}{s^2+4} \right) \right]$$

$$= -\frac{3}{5} \left[\frac{1}{3} \sin 3t - \frac{1}{2} \sin 2t \right]$$

Derivative of the Laplace :- مشتق لاپلاس

$$* n=1 \quad \mathcal{L}(t^n \cdot f(t)) = (-1)^n \frac{d^n}{ds^n} [\mathcal{L}(f(t))]$$

$$\mathcal{L}(t \cdot f(t)) = - \frac{d}{ds} [F(s)] \quad \text{یعنی } \mathcal{L}^{-1} \text{ پر کیجیے} \leftarrow$$

$$\rightarrow t \cdot f(t) = - \mathcal{L}^{-1} \left(\frac{d}{ds} (F(s)) \right)$$

$$\left| \mathcal{L}^{-1} \left(\frac{d}{ds} (F(s)) \right) = -t \cdot f(t) \right| \text{ Important.}$$

$$* n=2 \quad \mathcal{L}(t^2 \cdot f(t)) = \frac{d^2}{ds^2} [F(s)]$$

$$\left| \mathcal{L}^{-1} \left(\frac{d^2}{ds^2} (F(s)) \right) = t^2 \cdot f(t) \right| \text{ Important}$$

$$\# \mathcal{L}^{-1} \left(\frac{d^n}{ds^n} (F(s)) \right) = (-1)^n t^n \cdot f(t) \quad \# \quad \text{--- s plz d R w *}$$

Ex 3- $\mathcal{L}^{-1}\left(\ln\left(\frac{s+1}{s+2}\right)\right)$

Sol:- $\mathcal{L}^{-1}\left(\ln(s+1) - \ln(s+2)\right)$

$\mathcal{L}^{-1}\left(\frac{d}{ds}(\ln(s+1) - \ln(s+2))\right) = -t \cdot f(t)$

$\mathcal{L}^{-1}\left(\frac{1}{s+1} - \frac{1}{s+2}\right) = -t \cdot \underline{f(t)}$ مشتق اللوغاريتم

$e^{-t} - e^{-2t} = -t \cdot f(t) \rightarrow f(t) = \frac{e^{-t}}{t} + \frac{e^{-2t}}{t}$

Ex 3- find $\mathcal{L}^{-1}\left(\tan^{-1}\left(\frac{1}{s}\right)\right)$

Sol:- $\mathcal{L}^{-1}\left(\frac{d}{ds}\left(\tan^{-1}\left(\frac{1}{s}\right)\right)\right) = -t \cdot f(t)$

$\mathcal{L}^{-1}\left(\frac{-\frac{1}{s^2}}{1 + \left(\frac{1}{s}\right)^2}\right) = -t \cdot f(t)$

$\mathcal{L}^{-1}\left(\frac{-1}{s^2 + 1}\right) = -t \cdot f(t)$

$\sin t = t \cdot \underline{f(t)}$ مشتق الجيب

$f(t) = \frac{\sin t}{t}$

Ex:- if inverse Laplace Transformation

$$\mathcal{L}^{-1}\left(\underbrace{\ln\left(\frac{s-1}{s+2}\right)}_{F(s)}\right) = f(t); \text{ find } \int_0^{\infty} e^{-6t} \cdot f(t) \cdot dt$$

Sol:-

$$\ln\left(\frac{6-1}{6+2}\right) = \ln\left(\frac{5}{8}\right)$$

Ex:- Given that $\mathcal{L}\left(\frac{1}{\sqrt{t}}\right) = \sqrt{\frac{\pi}{s}}$; find $\mathcal{L}\left(e^{-t} \cdot \frac{1}{\sqrt{t}}\right)$

Sol:- $\frac{1}{\sqrt{\pi}} \mathcal{L}\left(e^{-t} \cdot \frac{1}{\sqrt{t}}\right) = \frac{1}{\sqrt{\pi}} \cdot \sqrt{\frac{\pi}{s+1}} = \frac{1}{\sqrt{s+1}}$

* Unit Step Function:-

The unit step function $u(t)$ is defined by:-

$$u(t) = \begin{cases} 0, & t < 0. \\ 1, & t > 0. \end{cases}$$

$$\textcircled{1} \mathcal{L}(u(t-a) \cdot f(t-a)) = e^{-as} \cdot \mathcal{L}(f(t)).$$

$$\textcircled{2} \mathcal{L}(u(t-a)) = \frac{e^{-as}}{s}$$

$$\textcircled{3} \mathcal{L}(u(t-a) \cdot f(t)) = e^{-as} \cdot \mathcal{L}(f(t+a))$$

$$\text{Ex 3- } \mathcal{L}(u(t-3) \cdot \sin(t-3)) =$$

$$\text{Sol 3- } e^{-3s} \cdot \mathcal{L}(\sin t) = e^{-3s} \cdot \frac{1}{s^2+1} = \frac{e^{-3s}}{s^2+1}$$

$$\text{Ex 3- } \mathcal{L}(u(t-\frac{\pi}{2}) \cdot \sin t) = e^{-\frac{\pi}{2}s} \cdot \mathcal{L}(\sin(t+\frac{\pi}{2}))$$

$$\text{Sol 3- } e^{-\frac{\pi}{2}s} \cdot \mathcal{L}(\cos t) = e^{-\frac{\pi}{2}s} \cdot \frac{s}{s^2+1} = \frac{s \cdot e^{-\frac{\pi}{2}s}}{s^2+1}$$

$$\text{Ex 3- } \mathcal{L}(u(t-4) \cdot e^t) =$$

$$\text{Sol 3- } e^{-4s} \cdot \mathcal{L}(e^{t+4}) = e^{-4s} \cdot e^4 (\mathcal{L}(e^t))$$

$$= e^{-4s+4} \cdot \frac{1}{s-1} = \frac{e^{-4s+4}}{s-1}$$

$$\text{Ex 3- } \mathcal{L}(e^{2t} \cdot u(t-2)) =$$

$$\text{Sol 3- } e^2 \mathcal{L}(e^t \cdot u(t-2)) = e^2 \cdot e^{-2s} \mathcal{L}(e^{-(t+2)})$$

$$= e^{-2s} \frac{1}{s+1}$$

$$\text{Ex 3- } \mathcal{L}((t-1)^3 \cdot e^{-t} \cdot u(t-1)) =$$

$$\text{Sol 3- } e^{-s} \cdot \mathcal{L}(t^3 \cdot e^t) = e^{-s} \cdot \frac{3!}{(s-1)^4} = \frac{6 \cdot e^{-s}}{(s-1)^4}$$

Ex 3- $\mathcal{L}((3t+1) \cdot u(t-3))$ بنوع ال Unit تابع

$$\begin{aligned} \text{Sol- } & \mathcal{L}(3t \cdot u(t-3) + u(t-3)) \\ &= 3\mathcal{L}(t \cdot u(t-3)) + \mathcal{L}(u(t-3)) \\ &= 3 \cdot e^{-3s} \cdot \mathcal{L}(t+3) + \frac{e^{-3s}}{s} = 3e^{-3s} \left(\frac{1}{s^2} + \frac{3}{s} \right) + \frac{e^{-3s}}{s} \\ &= e^{-3s} \left(\frac{3}{s^2} + \frac{3}{s} + \frac{1}{s} \right) = e^{-3s} \left(\frac{3}{s^2} + \frac{4}{s} \right). \end{aligned}$$

Ex 3- $f(t) = \begin{cases} 2, & t < 3 \\ -1, & t > 3 \end{cases}$ find $\mathcal{L}(f(t))$?

ملاحظة 3- اولا اشارة السالب -3

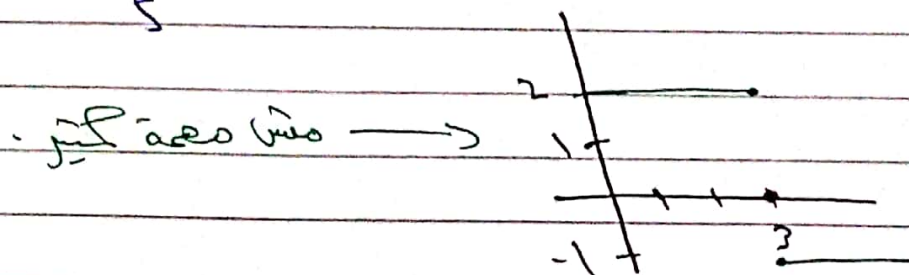
يجب إعادة تعريف $f(t)$ من جديد

(1) ننسب أول رقم الى صفر من ال 2

(2) ننزل الرقم الى تحت ونخرج منه الرقم الى فوقه وبتحويل ال Unit الى يكون علامة من نقطة ال jump

Sol- $2 - 3u(t-3) \rightarrow f(t) = 2 - 3u(t-3)$

$$\begin{aligned} \mathcal{L}(f(t)) &= \mathcal{L}(2 - 3u(t-3)) \rightarrow \mathcal{L}(2) - 3\mathcal{L}(u(t-3)) \\ &= \frac{2}{s} - 3 \cdot \frac{e^{-3s}}{s} \end{aligned}$$



Ex 3- $f(t) = \begin{cases} 3, & t < 2 \\ -1, & 2 < t < 5 \\ 7, & t > 5 \end{cases}$; find $\mathcal{L}(f(t))$?

Sol:- $f(t) = 3 - 4 \cdot u(t-2) + 8 \cdot u(t-5)$

$$\mathcal{L}(f(t)) = \mathcal{L}(3 - 4u(t-2) + 8u(t-5))$$

$$\mathcal{L}(3) - 4\mathcal{L}(u(t-2)) + 8\mathcal{L}(u(t-5)) = \frac{3}{s} - 4 \cdot \frac{e^{-2s}}{s} + 8 \cdot \frac{e^{-5s}}{s}$$

Ex 3- $f(t) = \begin{cases} e^t, & 0 < t < 2\pi \\ e^t + \cos t, & t > 2\pi \end{cases}$; find $\mathcal{L}(f(t))$?

Sol:- $f(t) = e^t + (\cos t) \cdot u(t-2\pi)$

$$\mathcal{L}(f(t)) = \mathcal{L}(e^t + \cos t \cdot u(t-2\pi))$$

$$= \mathcal{L}(e^t) + \mathcal{L}(u(t-2\pi) \cdot \cos t)$$

$$= \frac{1}{s-1} + e^{-2\pi s} \cdot \mathcal{L}(\cos(t+2\pi)) = \frac{1}{s-1} + e^{-2\pi s} \cdot \mathcal{L}(\cos t)$$

$$= \frac{1}{s-1} + e^{-2\pi s} \cdot \frac{s}{s^2+1}$$

* Inverse Unit step function

$$\textcircled{1} \mathcal{L}^{-1}\left(\frac{e^{-as}}{s}\right) = u(t-a).$$

$$\textcircled{2} \mathcal{L}^{-1}(e^{-as} \cdot F(s)) = u(t-a) \cdot f(t-a).$$

$$\text{Ex 3- Find } \mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^3}\right)$$

$$\text{Sol 3- } u(t-2) \cdot \frac{1}{2} \mathcal{L}^{-1}\left(\frac{2}{s^3}\right) = \frac{1}{2} u(t-2) \cdot (t-2)^2$$

• لازم توجه قوس (t)

$$\text{Ex 3- } \mathcal{L}^{-1}\left(\frac{e^{-\pi s}}{s^2+1}\right)$$

$$\text{Sol 3- } u(t-\pi) \cdot \sin(t-\pi)$$

$$\text{Ex 3- } \mathcal{L}^{-1}\left(\frac{e^{-3s} \cdot s}{(s^2+4)}\right)$$

$$\text{Sol 3- } u(t-3) \cdot \cos 2(t-3)$$

$$\text{Ex 3- } \mathcal{L}^{-1}\left(\frac{e^{-6s}}{(s-4)^3}\right)$$

$$\text{Sol 3- } \frac{1}{2} \mathcal{L}^{-1}\left(\frac{e^{-6s} \cdot 2}{(s-4)^3}\right) = u(t-6) \frac{1}{2} \cdot e^{u(t-6)} \cdot (t-6)^2$$

$$= \frac{1}{2} u(t-6) \cdot e^{u(t-6)} \cdot (t-6)^2$$

$$\text{Ex 3- } \mathcal{L}^{-1}\left(\frac{e^{-4s}}{s^2-2s-3}\right)$$

$$\text{Sol 3- } u(t-4) \cdot \mathcal{L}^{-1}\left(\frac{1}{s^2-2s-3}\right) \quad \leftarrow \text{نقل ونحدد القوس}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2-2s+3}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2-2s-3+1-1}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{1}{s^2-2s+1-4}\right) = \frac{1}{2} \mathcal{L}^{-1}\left(\frac{2}{(s-1)^2-4}\right)$$

$$= \frac{1}{2} \mathcal{L}^{-1}\left(\frac{2}{(s-1)^2-4}\right) = \frac{1}{2} e^t \cdot \sinh 2t$$

$$= \frac{1}{2} u(t-4) \cdot e^{(t-4)} \cdot \sinh 2(t-4)$$

$$\text{Ex 3- } \mathcal{L}^{-1}\left(\frac{e^{-6s} \cdot 3}{(s-\pi)^2+9}\right)$$

$$\text{Sol 3- } u(t-6) \cdot e^{\pi(t-6)} \cdot \sin 3(t-6)$$

$$\mathcal{L}^{-1}\left(\frac{3}{(s-\pi)^2+9}\right) = e^{\pi t} \cdot \sin 3t$$

* Solving Initial value problems.

• To solve $a(x)y'' + b(x)y' + c(x)y = f(x)$
subjected to $y(a) = b, y'(a) = c$:-

① Take $\mathcal{L}$ to both side.

② Find $\mathcal{L}(y(t)) = Y(s)$.

③ Take $\mathcal{L}^{-1}$.

$$* \mathcal{L}(y^{(n)}(t)) = s^n \mathcal{L}(y(t)) - s^{n-1} y(0) - \dots - y^{(n-1)}(0)$$

$$1] \mathcal{L}(y'(t)) = sY(s) - y(0)$$

$$2] \mathcal{L}(y''(t)) = s^2 Y(s) - sy(0) - y'(0)$$

$$3] \mathcal{L}(y'''(t)) = s^3 Y(s) - s^2 y(0) - sy'(0) - y''(0)$$

Ex-3 Use Laplace Transform to solve:-

$$y' - 3y = e^{2t} \quad , \quad y(0) = 1 \quad ?!$$

$$\text{Sol: } \mathcal{L}(y' - 3y) = \mathcal{L}(e^{2t})$$

$$\mathcal{L}(y'(t)) - 3\mathcal{L}(y(t)) = \frac{1}{s-2}$$

$$sY(s) - y(0) - 3Y(s) = \frac{1}{s-2}$$

$$sY(s) - 1 - 3Y(s) = \frac{1}{s-2} \rightarrow sY(s) - 3Y(s) = \frac{1}{s-2} + 1$$

$$Y(s)[s-3] = \frac{1}{s-2} + 1 \rightarrow Y(s) = \frac{1}{(s-2)(s-3)} + \frac{1}{s-3}$$

$$\mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{1}{(s-2)(s-3)}\right) + \mathcal{L}^{-1}\left(\frac{1}{s-3}\right)$$

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{s-3} - \frac{1}{s-2}\right) + e^{3t}$$

$$= e^{3t} - e^{2t} + e^{3t} = 2e^{3t} - e^{2t}$$

Ex 3 - $y'' - 6y' + 13y = 0$, $y(0) = 0$, $y'(0) = 3$?!

Sol 3 - $\mathcal{L}(y'' - 6y' + 13y) = \mathcal{L}(0)$

$$\mathcal{L}(y'') - 6\mathcal{L}(y') + 13\mathcal{L}(y) = 0$$

$$s^2 Y(s) - sY(0) - Y'(0) - 6(sY(s) - Y(0)) + 13Y(s) = 0$$

$$s^2 Y(s) - 0 + 3 - 6(sY(s)) + 13Y(s) = 0$$

$$Y(s) [s^2 - 6s + 13] = 3 \rightarrow Y(s) = \frac{3}{s^2 - 6s + 13}$$

$$\mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{3}{s^2 - 6s + 13}\right)$$

$$y(t) = \mathcal{L}^{-1}\left(\frac{3}{s^2 - 6s + 13 + 9 - 9}\right)$$

$$y(t) = \mathcal{L}^{-1}\left(\frac{3}{s^2 - 6s + 9 + 4}\right) = \mathcal{L}^{-1}\left(\frac{3}{(s-3)^2 + 4}\right)$$

$$y(t) = \frac{2}{4} \mathcal{L}^{-1}\left(\frac{2}{(s-3)^2 + 4}\right) = \frac{1}{2} e^{3t} \sin 2t$$