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Continuous change

دفتر :

معادلات تفاضلية عادية

Ordinary Differential Equations

للدكتور : فراس

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اللجنة الأكاديمية لقسم الهندسة الصناعية

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Differential Equations

Chapter one: Introduction

Derivative

Ordinary derivative
المشتقة العادية

* Contains only 1 Independent Variable
تحتوي على متغير مستقل واحد

* Contains 1 or more dependent Variable
تحتوي على متغير تابع واحد أو اكثر

Ex) $y = 3x^2 + \sin x$
find y'.

~~y: [illegible]dependent~~
~~x: dependent~~

y: dependent
x: Independent

$y' = 6x + \cos x$

~~y': Independent Variable~~
~~x: Dependent Variable~~

y': dependent V.
x: Independent

Partial derivative
المشتقة الجزئية

* contains more than 1 Independent Variable
تحتوي على اكثر من متغير مستقل واحد

Ex) $u = 2xy - 3x^2 + y$
find

① $u_x = \frac{\partial u}{\partial x} = 2y - 6x$

② $u_y = \frac{\partial u}{\partial y} = 2x + 1$

③ $u_{xy} = \frac{\partial^2 u}{\partial y \partial x} = 2 - 0 = 2$

x: Independent V.
y: Independent V.
u: Dependent V.

التعريف: أي معادلة تحتوي على مشتقة

dependent V.

التعريف: أي معادلة تحتوي على مشتقات لمعامل تابع واحد أو اكثر مع وجود عامل مستقل واحد أو اكثر تسمى معادلة تفاضلية (D.E.) .

ملاحظة: مش ضروري تكون اكثر من مشتقة ⇐ واحدة أو اكثر.

Ex)

مشتقات

x: Independent V.

① $y'' + y' = 5x \longrightarrow$ D.E.

y: Dependent V.

Independent

② $\frac{\partial u}{\partial x} + 2x = u \longrightarrow$ D. E.

مشتقة — Dependent V.

الخاصية الخطية الرتبة النوع

D.E.s are classified by: Type, Order, and Linearity

I) Classification by type:

Ordinary differential Equation (O.D.E)

Ex) $\frac{dy}{dx} = 3xy$

Partial differential Equation (PDE)

Ex) $\frac{\partial u}{\partial x} + \frac{2\partial u}{\partial z} = 2xz$

Ex)

① $y' + y'' = 2x\ln y \longrightarrow$ Ordinary DE

② $\frac{d^2y}{dx^2} + 5y = e^x \longrightarrow$ ordinary DE

③ $u_{xy} + 2u_x = 3x \longrightarrow$ Partial DE

④ $\frac{\partial u}{\partial x} + 2xu = 3 \longrightarrow$ Partial DE

⑤ $(\theta'')^2 + 5\theta' = \theta \longrightarrow$ Ordinary DE

ⅠⅠ Classification by order:

الرتبة للمعادلة التفاضلية تكون رتبة أعلى مشتقة في المعادلة

Ex) What is the order for each of the following equation:

① $y'' - y' \cdot y'' - 5x = 0 \longrightarrow$ Order $= 2$

② $(y^{(5)})''' - y^{(6)} - 2 \ln x = 7 \longrightarrow$ order $= 8$

③ $(y^{(3)})^5 + 2y'' + y' = 2 \longrightarrow$ order $= 3$

④ $\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x \partial z} + 5 \longrightarrow$ order $= 2$

\boxed{III} Classification as linear or Non-Linear

كي تكون المعادلة خطية يجب أن :

Dependent V.

١- تكون القوة لجميع مشتقات المتغير التابع تساوي واحد = ١ .

٢- لا يكون المتغير التابع مضروب بنفسه أو ب[illegible] بأحد مشتقاته .

٣- أن لا يكون المتغير التابع تحت جذر أو في المقام أو داخل Ln أو داخل أسي (e^y) أو داخل اقتران مثلثي ($\sin y$)

٤- معاملات المتغير التابع تعتمد على (x) أو على ثابت (طبعًا على اعتبار أن (y) هي المتغير التابع و(x) المتغير المستقل)

The n^{th} order, linear DE has the form:

$$a_n(x)\,y^{(n)} + a_{n-1}(x)\,y^{(n-1)} + \dots + a_1(x)\,y' + a_0(x)\,y = g(x)$$

Ex)
① $y'' + 2xy' + 3x^2y = e^x \longrightarrow$ linear in y
② $3r\theta' + 2\theta = r \longrightarrow$ linear in θ
③ $3yx'' + 2x''' = 2x \longrightarrow$ linear in x
④ $2xy'' + y' \cdot y^{(3)} + 2xy = Z \longrightarrow$ Non-Linear
⑤ $2xy' + 2\sin(xy) = e^x \longrightarrow$ Non-Linear
⑥ $\frac{dy}{dx} = \frac{3}{x-e^y} \longrightarrow$ Non-Linear in y but linear in x because $\frac{dx}{dy} = \frac{x-e^y}{3}$

غير خطي عند (y) لكن اذا قلبنا المعادلة يصير خطي عند (x)

Solution of DE

Explicit Solution

Implicit solution (ضمني)

Ex) $y = f(x)$

Ex) $x^2 + y^2 = 4$ علاقة ضمنية

Ex) Show that $y = e^{2x}$ is an explicit solution to $y' - 2y = 0$

Solution: $y = e^{2x}$, $y' = 2e^{2x}$

نعوض في $y'-2y=0$ $2e^{2x} - 2e^{2x} = 0 \Rightarrow 0 = 0$ #

Ex) Show that $4x^2 - y^2 = 4$ is an implicit Solution to $y \cdot \frac{dy}{dx} - 4x = 0$

Solution: $4x^2 - y^2 = 4$, $8x - 2yy' = 0$

$$\boxed{y' = \frac{4x}{y}}$$

$\Rightarrow \cancel{y} \cdot \frac{4x}{\cancel{y}} - 4x = 0 \Rightarrow 0 = 0$ #

Theorem: consider the following IVP:

$$* \; y' + P(x)\,y = g(x) \quad \text{where } y(x_0) = y_0$$

If $P(x)$ and $g(x)$ are continuous on (a,b) that contains x_0, then there is a unique solution for the Initial Value Problem (IVP) on the interval (a,b)

Ex) find the largest open interval where the following IVP has a unique solution

① $(x^2-9)\,y' + 2y = \ln|20-4x| \;,\; y(4) = -3$

لازم نحول هذه المعادلة للصورة الأصلية *

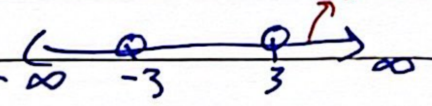
Sol: Divide by (x^2-9)

حصلنا على الصورة الأصلية ← $y' + \dfrac{2}{x^2-9}\,y = \dfrac{\ln|20-4x|}{x^2-9}$

نستخرج $P(x)$ معامل (y) و $g(x)$ بعد المساواة و نجد مجال كل منهما ثم تقاطعهم ببعض ← $P(x) = \dfrac{2}{x^2-9}$

$* \; x^2-9=0$

$x = \pm 3$



$P(x) \in (3,\infty)$

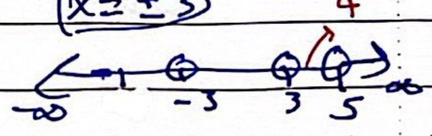
$g(x) = \dfrac{\ln|20-4x|}{x^2-9}$

$* \; 20-4x > 0$

$x<5 \qquad (-\infty,5)$

$* \; x^2-9=0$

$\boxed{x=\pm 3}$



$\therefore$ The largest interval is:

$(3,\infty) \cap (3,5) \boxed{= (3,5)}$

$g(x) \in (3,5)$

② $(x-9)\,y' + \frac{2x}{\ln(x-5)}\,y = \frac{3}{x}$, $y(7) = 3$

Solution: $y' + \frac{2x}{(x-9)\ln(x-5)}\,y = \frac{3}{(x-9)x}$

$P(x) = \frac{2x}{(x-9)\ln(x-5)}$

* $x-9=0$

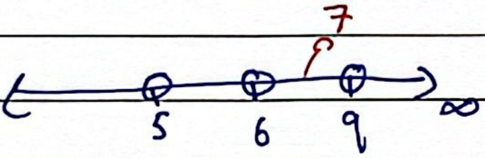
$\boxed{x=9}$

* $\ln(x-5)=0$

$\boxed{x=6}$

* $x-5>0$

$x>5$ $(5,\infty)$



$P(x) \in (6,9)$

$g(x) = \frac{3}{(x-9)x}$

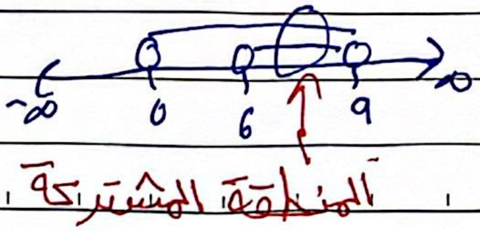
* $x-9=0$

$\boxed{x=9}$

* $\boxed{x=0}$

$g(x) \in (0,9)$

∴ The largest interval is : $(6,9) \cap (0,9)$



$\boxed{=(6,9)}$

Chapter 2: First order DE

This DE has the form $\frac{dy}{dx} = f(x,y)$

Separable Equation:

This DE has the form $\frac{dy}{dx} = g(x) \cdot h(y)$

Ex) Determine the separable solution from the following:

① $\frac{dy}{dx} = e^{x-3y}$

Sol: $\frac{dy}{dx} = e^{x} \cdot e^{-3y}$ Separable

② $\frac{dy}{dx} = \frac{xy+y}{yx+x}$

Sol: $\frac{dy}{dx} = \frac{y(x+1)}{x(y+1)} \Rightarrow \frac{dy}{dx} = \frac{y}{y+1} \cdot \frac{x}{x+1}$ —— Separable

③ $\frac{dy}{dx} = x \sin(x+y)$

Not separable

* To solve the Separable equation $\frac{dy}{dx} = g(x) \cdot h(y)$

$1 \Rightarrow \frac{1}{h(y)} dy = g(x)\, dx$

$2 \Rightarrow \int \frac{1}{h(y)} dy = \int g(x)\, dx$

Ex) Solve : $(2+x) \frac{dy}{dx} = (y+3)$ → Separable

Sol: $(2+x) \frac{dy}{dx} = (y+3)$

$\Rightarrow (2+x)\, dy = (y+3)\, dx$

$\Rightarrow \frac{1}{y+3} dy = \frac{1}{2+x} dx$

$\Rightarrow \int \frac{1}{y+3} dy = \int \frac{1}{2+x} dx$

$\Rightarrow \boxed{\ln|y+3| = \ln|2+x| + c}$ → Implicit Soluti

If I want an Explicit Solution :

$|y+3| = e^{\ln|2+x| + c}$ $\Rightarrow |y+3| = |2+x| \cdot e^{c}$

$\Rightarrow |y+3| = |2+x| \cdot c$ $\Rightarrow y+3 = \pm |2+x| \cdot c$

$\Rightarrow y = -3 \pm |2+x| \cdot c$

$\Rightarrow \underbrace{\boxed{y = -3 + c|2+x|}}_{\text{Explicit Solution}}$ and $\underbrace{\boxed{y = -3 - c|2+x|}}_{\text{Explicit Sol.}}$

Ex) Solve:

$$e^{x} y \frac{dy}{dx} = e^{-y} + e^{-2x-y}$$

Sol:

e^{-x} نضرب $\Rightarrow y\frac{dy}{dx} = e^{-x}\cdot e^{-y} + e^{-3x}\cdot e^{-y}$

e^{y} نضرب $\Rightarrow y e^{y}\frac{dy}{dx} = e^{-x} + e^{-3x}$

$\Rightarrow y e^{y} dy = e^{-x} + e^{-3x} \cdot dx$ ... Separable

$\int$ نركب تكامل $\Rightarrow \int \underbrace{y e^{y}}\, dy = \int e^{-x} + e^{-3x} \cdot dx$

$$\Rightarrow \boxed{y e^{y} - e^{y} = -e^{-x} - \frac{e^{-3x}}{3} + C}$$

لا تنسى

Implicit Solution

كامل		اشتق
e^{y}	$+$	y
e^{y}	$-$	1
e^{y}		0

Ex) Solve $\frac{dy}{dx} = \frac{xy + 3x + 2y + 6}{xy - 2y + 4x - 8}$

<u>Sol:</u>

$$\frac{dy}{dx} = \frac{x(y+3) + 2(y+3)}{y(x-2) + 4(x-2)}$$

$$\frac{dy}{dx} = \frac{(y+3)\,(x+2)}{(x-2)\,(y+4)}$$

separable $\left[\frac{dy}{dx} = \frac{(y+3)}{(y+4)} \cdot \frac{(x+2)}{(x-2)}\right. \Rightarrow \frac{(y+4)}{(y+3)}\,dy = \frac{(x+2)}{(x-2)}\,dx$

$$\int \frac{y+4}{y+3}\,dy = \int \frac{x+2}{x-2}\,dx$$

$$\int 1 + \frac{1}{y+3}\,dy = \int 1 + \frac{4}{x-2}\,dx$$

$$\boxed{\Rightarrow y + \ln|y+3| = x + 4\ln|x-2| + C}$$

$\downarrow$

Implicit Solution

$$\begin{array}{r|l} & 1 \\ x-2 & x+2 \\ & \mp x \pm 2 \\ \hline & 4 \end{array}$$

$$\begin{array}{r|l} & 1 \\ y+3 & y+4 \\ & \mp y \mp 3 \\ \hline & 1 \end{array}$$

Ex) find the values of M and N which make the D.E. separable.

لاحظ السؤال يتعلق انواع separable

$$\frac{dy}{dx} = \frac{xy + 2y + 3x + M^2}{xy - 2x + 4y - n}$$

Sol: $$\frac{dy}{dx} = \frac{y(x+2) + 3\left(x + \frac{M^2}{3}\right)}{x(y-2) + 4\left(y - \frac{n}{4}\right)}$$

$$\frac{m^2}{3} = 2 \Rightarrow m^2 = 6$$

$$\boxed{M = \pm\sqrt{6}}$$

$$+2 = +\frac{N}{4} \Rightarrow \boxed{N = 8}$$

Ex) Solve the IVP: $(x+1)dy - (y^2+1)dx = 0$, $y(0) = 1$

Sol: $(x+1)dy = (y^2+1)dx$

separable ← $\frac{1}{y^2+1}dy = \frac{1}{x+1}dx \Rightarrow \int \frac{1}{y^2+1}dy = \int \frac{1}{x+1}dx$

$$\tan^{-1} y = \ln|x+1| + c$$

Since $y(0) = 1 \Rightarrow \tan^{-1} 1 = \ln|1| + c$

$\overset{\swarrow}{x}$ $\overset{\swarrow}{y}$

$$\frac{\pi}{4} = 0 + c \qquad \boxed{\therefore\ c = \frac{\pi}{4}}$$

$$\boxed{\begin{aligned} &\therefore \tan^{-1} y = \ln|x+1| + \frac{\pi}{4} \\ &\therefore y = \tan\left(\ln|x+1| + \frac{\pi}{4}\right) \end{aligned}}$$

Ex) If $f'(t) = \frac{f^2(t)+1}{t+e}$ with $f(0)=0$

find $f(2) = ?$

Sol:

$$f'(t) = \frac{df}{dt} \quad \Rightarrow \frac{df}{dt} = \frac{f^2(t)+1}{t+e}$$

$$\Rightarrow \frac{1}{f^2(t)+1}\, df = \frac{1}{t+e}\, dt$$

$$\Rightarrow \int \frac{1}{f^2(t)+1}\, df = \int \frac{1}{t+e}\, dt$$

$$\tan^{-1}(f) = \ln|t+e| + c$$

Since $f(\underset{t}{0}) = \underset{f}{0}$ $\Rightarrow \tan^{-1}(0) = \ln|0+e| + c$

$$0 = 1 + c$$

$$\boxed{c = -1}$$

$$\therefore \tan^{-1}(f) = \ln|t+e| - 1$$

$$\therefore f(t) = \tan(\ln|t+e| - 1)$$

$$\boxed{\therefore f(2) = \tan(\ln|2+e| - 1)}$$

Linear Equation in first Order

* The Standard form of this equation is:

Linear in y ← $$y' + P(x)\,y = g(x) \quad — (*)$$

معامل y' يجب أن يكون 1

* To Solve this equation:

① Find the Integrating factor ($\mu(x)$) as:

$$\mu(x) = e^{\int P(x)\,.dx}$$

② Multiply both sides of equation (*) by $\mu(x)$, then we have: $\mu(x)\left[y' + P(x)y = g(x)\right]$

$$\underbrace{\mu(x)\,y' + \mu(x)\,P(x)\,y}_{\text{This side equals } \frac{d}{dx}[\mu(x)\,y]} = \mu(x)\,g(x)$$

③ Integrate both sides (with respect x)

$$\int \frac{d}{dx}\left[\mu(x)\,y\right] = \int \mu(x)\,g(x)\,dx$$

$$\mu(x)\,y = \int \mu(x)\,g(x)\,.dx + c$$

$$\boxed{y = \frac{1}{\mu(x)} \int \mu(x)\,g(x)\,dx + \frac{c}{\mu(x)}} \rightarrow \text{This is the Solution}$$

Remark:

$$\boxed{\mu(x) = e^{\int P(x)\,dx}}$$

① $\mu'(x) = P(x) \cdot e^{\int P(x)\,dx}$

$= P(x) \cdot \mu(x)$

② $P(x) = \frac{\mu'(x)}{\mu(x)}$

③ $\frac{d}{dx}[\mu(x)\, y] = \mu'(x)\, y + \mu(x)\, y$

$= P(x)\mu(x) y + \mu(x)\, y'$

Ex) Solve $(x^2+1)\frac{dy}{dx} + xy - x = 0$

Sol: $(x^2+1)y' + xy = x$

$\Rightarrow y' + \frac{x}{x^2+1} y = \frac{x}{x^2+1}$ — Linear in y

* $P(x) = \frac{x}{x^2+1}$, $g(x) = \frac{x}{x^2+1}$

* The Integrating factor is:

$\mu(x) = e^{\int \frac{x}{x^2+1}\,dx} \Rightarrow e^{\frac{1}{2}\int \frac{2x}{x^2+1}\,dx}$

$\Rightarrow e^{\frac{1}{2}\ln|x^2+1|} \Rightarrow \mu(x) = (x^2+1)^{\frac{1}{2}}$

* The Solution: $y = \frac{1}{\mu(x)} \int \mu(x)\, g(x)\,dx + \frac{c}{\mu(x)}$

$y = \frac{1}{(x^2+1)^{\frac{1}{2}}}$ $y = \frac{1}{(x^2+1)^{\frac{1}{2}}} \int (x^2+1)^{\frac{1}{2}} \left(\frac{x}{x^2+1}\right) dx + \frac{c}{(x^2+1)}$

$$\Rightarrow \frac{1}{(x^2+1)^{\frac{1}{2}}} \int \underbrace{x\,(x^2+1)^{-\frac{1}{2}}}_{\text{substitution}} dx + \frac{c}{(x^2+1)^{\frac{1}{2}}}$$

$$\Rightarrow \frac{1}{(x^2+1)^{\frac{1}{2}}} \int x\,(u)^{-\frac{1}{2}} \cdot \frac{du}{2x} + \frac{c}{(x^2+1)^{\frac{1}{2}}}$$

$$\boxed{\begin{aligned} u &= x^2+1 \\ du &= 2x \cdot dx \\ dx &= \frac{du}{2x} \end{aligned}}$$

$$\Rightarrow \frac{1}{(x^2+1)^{\frac{1}{2}}} \int \frac{1}{2}(u)^{-\frac{1}{2}} \cdot du + \frac{c}{(x^2+1)^{\frac{1}{2}}}$$

$$\Rightarrow \frac{1}{(x^2+1)^{\frac{1}{2}}} \left(u^{\frac{1}{2}}\right) + \frac{c}{(x^2+1)^{\frac{1}{2}}}$$

$$\Rightarrow \frac{1}{\cancel{(x^2+1)^{\frac{1}{2}}}} \cancel{(x^2+1)^{\frac{1}{2}}} + \frac{c}{(x^2+1)^{\frac{1}{2}}}$$

$$\boxed{y = 1 + \frac{c}{\sqrt{x^2+1}}}$$

Ex) Solve $xy' - (x-1)y = x^2$

Sol: $y' - \frac{(x-1)}{x} y = x$ Linear in y

$* \; P(x) = \frac{-(x-1)}{x}$, $g(x) = x$

$P(x) = \frac{1}{x} - 1$

$* \; \mu(x) = e^{\int \frac{1}{x} - 1 \, dx}$ $\Rightarrow \boxed{\mu(x) = xe^{-}}$

* The Solution:

$$y = \frac{1}{\mu(x)} \int \mu(x) g(x) \, dx + \frac{c}{\mu(x)}$$

$$y = \frac{1}{xe^{-x}} \int \underline{x^2} e^{-x} \, dx + \frac{c}{xe^{-x}}$$

$$= \frac{1}{xe^{-x}} \left(-x^2 e^{-x} - 2x e^{-x} - 2e^{-x}\right) + \frac{c}{xe^{-x}}$$

/	S
x^2	$+ e^{-x}$
$2x$	$- e^{-x}$
2	$+ e^{-x}$
0	$- e^{-x}$

$$\Rightarrow \frac{1}{x\cancel{e^{-x}}} \cancel{e^{-x}} \left(-x^2 - 2x - 2\right) + \frac{c}{xe^{-x}}$$

$$\boxed{y = -x - 2 - \frac{2}{x} + \frac{c}{xe^{-x}}}$$

Ex) Solve the IVP

$$\frac{dy}{dx} = \frac{1}{e^y - x} \quad \text{where } y(1) = 0$$

Sol:

نقلب المعادلة لتصبح خطية $\Rightarrow \dfrac{dx}{dy} = \dfrac{e^y - x}{1}$

$$\Rightarrow x' = e^y - x \quad \Rightarrow x' + x = e^y$$

* $P(y) = 1$, $g(y) = e^y$

* $\mu(y) = e^{\int 1 . dy}$ $\Rightarrow \mu_y = e^y$

* Solution is:

$$y = \frac{1}{e^y} \int e^{2y} . dy + \frac{c}{e^y}$$

$$= \frac{1}{e^y} . \frac{e^{2y}}{2} + \frac{c}{e^y}$$

$$\boxed{x = \frac{e^y}{2} + \frac{c}{e^y}}$$

Since $y(\overset{x}{1}) = \overset{y}{0}$ $\Rightarrow 1 = \frac{1}{2} + c$

$\boxed{\therefore c = \frac{1}{2}}$ $\Rightarrow$ $\boxed{x = \frac{e^y}{2} + \frac{1}{2e^y}}$ or $\boxed{x = \frac{e^y + e^{-y}}{2} \quad x = \cosh y}$

Ex) Solve $(x+2)^2 y' = 5-4y-2xy$

Sol: $(x+2)^2 y' + \underbrace{4y+2xy} = 5$

$\Rightarrow (x+2)^2 y' + 2y(x+2) = 5$

$\Rightarrow (x+2)\ \left((x+2)y' + 2y\right) = 5$

$\Rightarrow y' + \frac{2y}{x+2} = \frac{5}{(x+2)^2}$

* $P(x) = \frac{2}{x+2}$, $g(x) = \frac{5}{(x+2)^2}$

* $\mu(x) = e^{\int \frac{2}{x+2} \cdot dx}$

$\Rightarrow \mu(x) = (x+2)^2$

* solution is: $y = \frac{1}{(x+2)^2} \int \cancel{(x+2)^2} \cdot \frac{5}{\cancel{(x+2)^2}} dx + \frac{c}{(x+2}$

$$\boxed{y = \frac{5x+c}{(x+2)^2}}$$

Ex) If $\mu(x) = 2^{\ln x}$ is an integrating factor for the linear DE: $xy' + 2x^2 P(x) y = x g(x)$ find $P(x)$.

<u>Sol:</u>

$$\boxed{\begin{gathered} y' + P(x) y = g(x) \\ P(x) = \frac{\mu'}{\mu} \end{gathered}}$$

قسمة على x $\Rightarrow y' + \underline{2x P(x)} y = g(x)$

$$2x P(x) = \frac{\mu'_x}{\mu}$$

$$2x P(x) = \frac{\frac{1}{x} \cdot \cancel{2^{\ln x}} \cdot \ln 2}{\cancel{2^{\ln x}}}$$

$$2x P(x) = \frac{\ln 2}{x} \Rightarrow \boxed{P(x) = \frac{\ln 2}{2x^2}}$$

Bernoulli Equation:

The standard form of this DE is:

$$y' + Q(x)\, y = f(x)\, y^n \;,\; n \in \mathbb{R}$$

Note that: ① If $n=0 \Rightarrow y' + Q(x)\,y = f(x) \rightarrow$ Linear in y

② If $n=1 \Rightarrow y' + Q(x)y = f(x)y.$

$\Rightarrow y' + y(Q(x) - f(x)) = 0 \rightarrow$ separable, Linear in y

الـLinear حالة خاصة
من الـBernoulli

* If $n \neq 0$ and $n \neq 1$, then we use $\boxed{V = y^{1-n}}$ to transform the D.E. from Bernoulli to Linear

STEPS:

1- Multiply the Bernoulli equation by y^{-n}, then we have

$$[y' + Q(x)\, y = f(x)\, y^n]\, y^{-n}$$

$$\Rightarrow \underline{y^{-n} y'} + Q(x)\, y^{1-n} = f(x)$$

2- $V = y^{1-n} \Rightarrow V' = (1-n)\, y^{-n} . y'$

$\Rightarrow \frac{V'}{1-n} = \underline{y^{-n} . y'} \quad \Rightarrow \frac{V'}{1-n} + Q(x) . V = f(x)$

نضرب $(1-n)$ $\Rightarrow V' + (1-n)\, Q(x) . V = (1-n) f(x) \rightarrow$ Linear in V

يوجد اختصار لهذه الخطوات ←

خطوات مختصرة ⇒ $y' + Q(x)\,y = f(x)\,y^{1-n}$ → Bernoulli in y

$$\Big\Downarrow \; V = y^{1-n}$$

$$\underbrace{V' + (1-n)\,Q(x)\,V = (1-n)\,f(x)}$$ → Linear in V

استخدم هذه المعادلة مباشرة

Ex) Solve $\frac{y'}{x} + y = y^2$

Sol:

نضرب (x) $\overset{V'}{y'} + \overset{(-1)}{x}\overset{V}{y} = \overset{(1-1)}{-x}y^2$ → Bernoulli in y

⇒ $V' - xV = -x$ → Linear in V

$\boxed{n = 2 \quad V = y^{1-n} = y^{-1}}$ (1-n)

$P(x) = -x \quad , \quad g(x) = -x$

$\mu(x) = e^{\int -x\,dx}$ ⇒ $\boxed{\mu(x) = e^{-\frac{x^2}{2}}}$

$V = \frac{1}{\mu(x)}\int \mu(x)\,g(x)\,dx + c$

⇒ $V = \frac{1}{e^{-\frac{x^2}{2}}}\int e^{-\frac{x^2}{2}} \cdot -x\,dx + \frac{c}{e^{-\frac{x^2}{2}}}$

⇒ $\frac{1}{e^{-\frac{x^2}{2}}}\int e^{u} \cdot +\cancel{x} \cdot +\frac{du}{\cancel{x}} + \frac{c}{e^{-\frac{x^2}{2}}}$

$\boxed{u = -\frac{x^2}{2} \\ \cancel{2}du = -\cancel{2}x\cdot \\ dx = -\frac{du}{x}}$

$V = \frac{1}{\cancel{e^{-\frac{x^2}{2}}}} \cdot \cancel{e^{-\frac{x^2}{2}}} + \frac{c}{e^{-\frac{x^2}{2}}}$

Solution of linear ← $\boxed{V = 1 + \frac{c}{e^{-\frac{x^2}{2}}}}$ but $V = y^{-1} = \frac{1}{y}$

∴ $\frac{1}{y} = 1 + \frac{c}{e^{-\frac{x^2}{2}}}$ ⇒ →

$\boxed{\therefore\ y = \dfrac{1}{1+C\cdot e^{\frac{x^2}{2}}}}$ → Solution of Bernoulli

Ex) Solve $xy' + y = \ln x\, y^2$

Sol:

قسمة على x $\quad y' + \frac{1}{x}y = \frac{\ln x\, y^2}{x}$ → Bernoulli in y

$n=2$
$v = y^{1-n}$
$v = y^{-1}$

$v' - \frac{1}{x}v = -\frac{\ln x}{x}$ → Linear in v

$p(x) = \frac{-1}{x} \quad , \quad g(x) = \frac{-\ln x}{x}$

$\mu(x) = e^{\int -\frac{1}{x}\cdot dx} \qquad \boxed{\mu(x) = \frac{1}{x}}$

$v = \frac{1}{\frac{1}{x}} \int \frac{1}{x} \cdot \frac{-\ln x}{x}\, dx + \frac{c}{\frac{1}{x}}$

$= x \int \frac{-\ln x}{x^2}\, dx + cx$

$= x\left(\frac{\ln x}{x} - \int \frac{1}{x^2}\, dx\right) + cx$

$u = \ln x \;,\; dv = \frac{-1}{x^2}$
$du = \frac{1}{x}dx \;,\; v = \frac{1}{x}$

$= x\left(\frac{\ln x}{x} + \frac{1}{x}\right) + cx$

$v = (\ln x + 1) + cx$

but $v = \frac{1}{y}$

$\frac{1}{y} = (\ln x + 1) + cx \Rightarrow \boxed{y = \frac{1}{(\ln x + 1) + cx}}$

Ex) Solve $\dfrac{dy}{dx} = \dfrac{1}{\frac{x}{y} + y x^2}$

Sol: $\dfrac{dx}{dy} = \dfrac{x}{y} + y x^2$

$\Rightarrow x' - \frac{1}{y} x = y x^2$ — Bernoulli in x

$\Rightarrow V' + \frac{1}{y} V = -y$ — Linear in V

$n = 2$
$V = x^{1-n}$
$V = x^{-1}$

$P(y) = \frac{1}{y}$, $g(y) = -y$

$\mu(y) = e^{\int \frac{1}{y} \cdot dy} \Rightarrow \mu(y) = y$

$V = \frac{1}{y} \int -y^2 \cdot dy + \frac{c}{y}$

$V = \frac{1}{y} \left(\frac{-y^3}{3} \right) + \frac{c}{y}$

$V = \frac{-y^3}{3} + \frac{c}{y}$

but $V = x^{-1}$

$$\boxed{\frac{1}{x} = \frac{-y^2}{3} + \frac{c}{y}}$$

لكن تركها بهذه الصورة
أو نوحد المقامات ونقلب

من هنا يصبح الاشتقاق الجزئي مطلوب

Review about Partial derivative:

Ex) $P(x,y) = x^2 + xy + \sin(2x+3y) - y + e^{xy}$

find

① $\underset{\frac{\partial P}{\partial x}}{\underset{\Downarrow}{P_x}} = 2x + y + 2\cos(2x+3y) + y\,e^{xy}$

② $\underset{\frac{\partial P}{\partial y}}{\underset{\Downarrow}{P_y}} = x + 3\cos(2x+3y) - 1 + x\,e^{xy}$

There is also Partial Integration

Ex) ① $\int (x e^{y} + xy + \sin(x+2y) + e^{xy})\,dx$

$= \frac{x^2}{2}e^{y} + \frac{x^2}{2}y - \cos(x+2y) + \frac{e^{xy}}{y} + g(y)$

② $\int (x e^{y} + yx + \sin(x+2y) + e^{xy})\,dy$

$= xe^{y} + \frac{y^2}{2}x - \frac{\cos(x+2y)}{2} + \frac{e^{xy}}{x} + h(x)$

Exact Equation

The DE $M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be exact if $M_y = N_x$ or $\left(\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\right.$

Ex) Determine whether the DE is exact or not.

① $\underbrace{(5x+4y)}_{M}dx + \underbrace{(4x-8y^3)}_{N}dy = 0$

$M_y = 4 \qquad N_x = 4$

$\therefore$ Since $M_y = N_x \Rightarrow$ Exact

② $(\underbrace{\sin y - x\cos y}_{M})\,dx + (\underbrace{x\cos y - \sin y}_{N})\,dy = 0$

$M_y = \cos y + x\sin y \qquad N_x = \cos y$

Since $M_y \neq N_x \Rightarrow$ Not Exact

Remark: If $f(x,y) = c$ is a solution of the exact DE: $M\,dx + N\,dy = 0$, then $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$

Ex) Solve $\underbrace{(\cos x \sin x - xy^2)}_{M}\,dx + \underbrace{(y - yx^2)}_{N}\,dy = 0$

Sol:

$M_y = 0 - 2xy \quad , \quad N_x = 0 - 2xy$

since $M_y = N_x \Rightarrow$ Exact

Suppose the solution is $f(x,y) = C$ such that:

الخطوات:

① $\frac{\partial f}{\partial x} = M$ and ② $\frac{\partial f}{\partial y} = N$

$\frac{\partial f}{\partial x} = \cos x \sin x - xy^2$

$\int \frac{\partial f}{\partial x}\,dx = \int (\cos x \sin x - xy^2)\,dx$

$f(x,y) = \int (\frac{1}{2}\sin(2x) - xy^2)\,dx$

$f(x,y) = \frac{-\cos 2x}{4} - \frac{x^2y^2}{2} + g(y)$

إذا كاملنا بالنسبة لـ (x) بنضيف اقتران (أو ثابت) بدلالة الـ (y) ⇐ g(y) اما اذا كاملنا بالنسبة لـ(y)

نشتق الاقتران اللي طلعناه بالخطوة الاولى f(x,y)

$\frac{\partial f}{\partial y} = y - yx^2$

$0 - x^2y + g'(y) = y - yx^2$

$g'(y) = y$

$\int g'(y)\,dy = \int y\,dy$

$g(y) = \frac{y^2}{2}$

$\therefore f(x,y) = \frac{-\cos(2x)}{4} - \frac{x^2y^2}{2} + \frac{y^2}{2}$

$\therefore$ The solution is:

$\boxed{\underline{-\cos(2x)} - \underline{x^2y^2} + \underline{y} = C}$

Ex) Solve $(y e^{xy} + y^2 - \frac{y}{x^2} + x)dx + (x e^{xy} + 2xy)dy$
$= (3 - \frac{1}{x})dy$

Sol: $\underbrace{(y e^{xy} + y^2 - \frac{y}{x^2} + x)}_{M} dx + \underbrace{(x e^{xy} + 2xy + \frac{1}{x} - 3)}_{N} dy = 0$

$$\left.\begin{aligned} M_y &= 1.e^{xy} + y x e^{xy} + 2y - \frac{1}{x^2} \\ N_x &= xy e^{xy} + e^{xy} + 2y - \frac{1}{x^2} \end{aligned}\right\} \Rightarrow M_y = N_x \quad \therefore \text{Exact}$$

∴ Suppose the solution is $f(x,y) = C$ such that:

$\frac{\partial f}{\partial x} = M$

$\frac{\partial f}{\partial x} = y e^{xy} + y^2 - \frac{y}{x^2} + x$

$\int \frac{\partial f}{\partial x} = \int (y e^{xy} + y^2 - \frac{y}{x^2} + x)dx$

$f(x,y) = e^{xy} + y^2 x + \frac{y}{x} + \frac{x^2}{2} + g(y)$

$\frac{\partial f}{\partial y} = N$

$\frac{\partial f}{\partial y} = \underbrace{x e^{xy} + 2xy + \frac{1}{x} - 3}$

$x e^{xy} + 2yx + \frac{1}{x} + g'(y) =$

نجعل الـ g'(y) موضوع قانون ثم نكامل للطرفين

$\cancel{x e^{xy}} + \cancel{2yx} + \cancel{\frac{1}{x}} + g'(y) = \cancel{x e^{xy}} + \cancel{2xy} + \cancel{\frac{1}{x}} - 3$

$\Rightarrow \boxed{g'(y) = -3} \overset{\text{نكامل}}{\Rightarrow} \boxed{\int g'(y)dy = \int -3dy}$

$\boxed{g(y) = -3y}$

$\therefore f(x,y) = e^{xy} + y^2 x + \frac{y}{x} + \frac{x^2}{2} - 3y$

$\boxed{\text{Solution: } e^{xy} + y^2 x + \frac{y}{x} + \frac{x^3}{2} - 3y = C}$

Ex) find the value of a and b which make the following DE exact

$$(\underbrace{xy^3+by^2}_{M})dx + (\underbrace{ax^2y^2+xy+y}_{N})dy = 0$$

Sol: Since the DE is exact

$$\Rightarrow M_y = N_x$$

$$\Rightarrow \underbrace{\boxed{3}xy^2}_{\textcircled{1}} + \underbrace{\boxed{2b}y}_{\textcircled{2}} = \underbrace{\boxed{2a}xy^2}_{\textcircled{1}} + \underbrace{\boxed{1}y}_{\textcircled{2}}$$

(+) او (-)
الآن نقارن المعاملات مع بعض (لا تنسى الاشارة)

① $3 = 2a \Rightarrow \boxed{a = \frac{3}{2}}$

② $2b = 1 \Rightarrow \boxed{b = \frac{1}{2}}$

Ex) If $\cos(xy) - 2x + \ln y = e^x - 3y^2 + 5$ is a solution for the exact DE $M\,dx + N\,dy = 0$ find the functions $M(x,y)$ and $N(x,y)$

Sol: يجب نقل جميع المتغيرات على جهة

$$\Rightarrow \underbrace{\cos(xy) - 2x + \ln y - e^x + 3y^2}_{f(x,y)} = 5$$

$$\therefore \; M = \frac{\partial f}{\partial x} \quad \text{and} \quad N = \frac{\partial f}{\partial y}$$

$$\Rightarrow \frac{\partial f}{\partial x} = -y\sin(xy) - 2 - e^x$$

$$M = -y\sin(xy) - 2 - e^x$$

and $\frac{\partial f}{\partial y} = -x\sin(xy) + \frac{1}{y} + 6y$

$$N = -x\sin(xy) + \frac{1}{y} + 6y$$

Five Apple

Remark: If $M\,dx + N\,dy = 0$ is exact eq., then

① $M(x,y) = \int N_x \, dy + h(x)$ لأن

② $N(x,y) = \int M_y \, dx + g(y)$

$$\boxed{\begin{array}{l}\text{Exact}\\ \Rightarrow M_y = N_x\end{array}}$$

Ex) find the general function $N(x,y)$ so that

$$\underbrace{\left(ye^{xy} + y^2 - \frac{y}{x^2}\right)}_{M}dx + N(x,y)\,dy = 0 \text{ is exact}$$

Sol:

Since the DE is exact

$$\Rightarrow M_y = N_x$$

$$e^{xy} + xy\,e^{xy} + 2y - \frac{1}{x^2} = N_x$$

مهم جدا

$$\Rightarrow \int N_x dx = \int\left(e^{xy} + \underbrace{xy\,e^{xy}}_{\text{اجزاء}} + 2y - \frac{1}{x^2}\right)dx + \boxed{g(y)}$$

$$N(x,y) = \frac{e^{xy}}{y} + xy\frac{e^{xy}}{y} - y\frac{e^{xy}}{y^2} + 2yx + \frac{1}{x} + g(y)$$

$$N(x,y) = \cancel{\frac{e^{xy}}{y}} + xe^{xy} - \cancel{\frac{e^{xy}}{y}} + 2xy + \frac{1}{x} + g(y)$$

L	∫ . dx
xy	e^{xy}
y	$\frac{e^{xy}}{y}$
0	$\frac{e^{xy}}{y^2}$

neral Fn. $\boxed{\therefore N(x,y) = xe^{xy} + 2xy + \frac{1}{x} + g(y)}$

where $N(1,y) = 5$ اذا اعطاني بالسؤال

$\Rightarrow N(1,y) = e^{y} + 2y + 1 + g(y) = 5$

اذا ما اعطاك قيمة
مشان تطلع $g(y)$ وكان
سؤال شو الفكرة
خد اللي بيكون فيه

Ex) Find the general function $M(x,y)$ so that the following DE is exact

$$\underbrace{(4M(x,y) \cancel{-}+ 2xy)}_{M}\,dx + \underbrace{(\sec^2 y + \frac{x}{y})}_{N}\,dy =$$

Sol:

exact $\Rightarrow$

$$\frac{\partial}{\partial y}(4M(x,y) + 2xy) = N_x$$

$$\frac{\partial}{\partial y}(4M(x,y) + 2xy) = \frac{1}{y}$$

$$\int \frac{\partial}{\partial y}(4M(x,y) + 2xy)\,dy = \int \frac{1}{y}\,dy$$

$$4M(x,y) + 2xy = \ln|y| + g(x)$$

نقسم على 4

$$\boxed{M(x,y) = \frac{-2xy + \ln|y| + g(x)}{4}}$$

H-w :

① find the Value of a and b so that the DE is exact

$$(ay^3 + bxy^4 - 2x)dx + (3xy^2 + 20x^2y^3)dy = 0$$

② If $xM(x,y) + yN(x,y) = 0$, find the solution of the DE $Mdx + Ndy = 0$

Soln.

① $M_y = 3ay^2 + 4bxy^3$

Exact $\Rightarrow M_y = N_x$

$N_x = 3y^2 + 40xy^3$

$\Rightarrow \boxed{3a}y^2 + \boxed{4b}xy^3 = \boxed{3}y^2 + \boxed{40x}y^3$

$3a = 3 \Rightarrow \boxed{a = 1}$ $\quad$ $4b = 40 \Rightarrow \boxed{b = 10}$

Five Apple

Special Integrated factor:

If the DE $M\,dx + N\,dy = 0$ isn't exact then: ⊛

① If the function $\boxed{\frac{M_y - N_x}{N}}$ depends only on x then the Special Integrated factor for eq. ⊛ is:

$$\mu(x) = e^{\int \frac{M_y - N_x}{N}\,dx}$$

Morever, the DE ⊛ becoms exact if it's multiplied by $\mu(x)$

② If the function $\frac{N_x - M_y}{M}$ depends on y only then the special integrating factor of DE ⊛ is:

$$\mu(y) = e^{\int \frac{N_x - M_y}{M}\,dy}$$

Morever, the DE ⊛ becoms exalt if it's multiplied by $\mu(y)$

① و ② مش طرق حل تستخدم اللي بدك إياه ، ① و ② هم حالات بتجرب الحالة ① وإذا ما زبط بتجرب الثانية (يعني بس حالة واحدة راح تزبط) حاول تعرف اي حالة من النظر أقرب

Ex) find the Integrating factor for the following non-exact DE

① $\underbrace{(x+y)}_{M} dx + \underbrace{x \ln x}_{N} dy = 0$

Sol:

$M_y = 1 \quad , \quad N_x = 1 + \ln x$

$M_y \neq N_x \Rightarrow$ Not exact

Now, $\frac{M_y - N_x}{N} = \frac{1 - \ln x - 1}{x \ln x} = \frac{-\ln x}{x \ln x} = \boxed{-\frac{1}{x}}$

يجب ان يكون ناتج الحالة الأولى يعتمد على (x) فقط، ان ظهر
(y) بتكون هذه الحالة [redacted] غير صحيحة

نكمل الحل ⇒ The special integrating factor is

$\mu(x) = e^{\int -\frac{1}{x} dx} = e^{-\ln x} = x^{-1} = \boxed{\frac{1}{x}}$

Ex) $\underbrace{6xy}_{M}\,dx + \underbrace{(4y+9x^2)}_{N}\,dy = 0$

Soln. :

$M_y = 6x \quad , \quad N_x = 18x$

$\Rightarrow M_y \neq N_x$, not exact

Now, $\dfrac{N_x - M_y}{M} = \dfrac{18x - 6x}{6xy} = \dfrac{\cancel{12}^{2}\cancel{x}}{\cancel{6}xy} = \boxed{\dfrac{2}{y}}$

يجب أن يكون جواب الحالة الثانية يعتمد على (y) فقط
اذا ظهر (x) بتكون هذه الحالة غير [redacted] صحيحة.

تكملة الحل $\Rightarrow$ The special I.F. is :

$\mu_{(y)} = e^{\int \frac{2}{y} dy} = e^{2\ln y} \boxed{= y^2}$

Ex) Solve the IVP

$\underset{M}{y}\,dx + (\underbrace{2x - ye^{y}}_{N})\,dy = 0$ where $y(1) = 0$

<u>Soln.</u>: $M_y = 1$, $N_x = 2$ $\Rightarrow M_y \neq N_x$

$\Rightarrow$ not exact

Now, $\frac{N_x - M_y}{M} = \frac{2-1}{y} = \boxed{\frac{1}{y}}$

يعتمد على
(y) فقط

$\therefore \mu(y) = e^{\int \frac{1}{y} dy} = e^{\ln y} \boxed{= y}$

Multiply the DE by $\mu(y) = y$:

$y\,(y\,dx + (2x - ye^{y})\,dy$

$\Rightarrow \underset{M}{y^2}\,dx + (\underbrace{2xy - y^2 e^{y}}_{N})\,dy = 0 \longrightarrow$ must be exact.

$\leftarrow$ check

$M_y = 2y$, $N_x = 2y$ $\Rightarrow M_y = N_x$, Exact ✓

The <u>soln.</u> is: $\frac{\partial f}{\partial x} = M$

$\Rightarrow \frac{\partial f}{\partial x} = y^2 \Rightarrow \int \frac{\partial f}{\partial x} dx = \int y^2 dx$

$\Rightarrow f(x,y) = y^2 x + g(y)$

$\frac{\partial f}{\partial y} = N \Rightarrow \cancel{2yx} + g'(y) = \cancel{2yx} - y^2$

$\Rightarrow \int g'(y)\,dy = \int -y^2 e^{y} dy$

$\boxed{g(y) = -y^2 e^{y} + 2ye^{y} - 2e^{y}}$

$\int d$

$-y^2 \quad + \quad e^{y}$

$-2y \quad \quad e^{y}$

Five Apple

$f(x,y) = y^2 x - e^y y^2 + 2y e^y - 2 e^y = c$

Since $y(\underset{x}{1}) = \underset{y}{0}$

$\Rightarrow 0 - 0 + 0 - 2 = c$

$\boxed{c = -2}$

$$\boxed{y^2 x - e^y y^2 + 2y e^y - 2 e^y = -2}$$

Ex) Solve $\underbrace{y}_{M}\,dx + \underbrace{(y^2 - x)}_{N}\,dy = 0$

Soln.: $M_y = 1$, $N_x = -1$ $\Rightarrow M_y \neq N_x$

Now, $\dfrac{N_x - M_y}{M} = \dfrac{-1-1}{y}$ $\boxed{= \dfrac{-2}{y}}$

تعتمد على (y) فقط ✓

$\Rightarrow \mu(y) = e^{\int \frac{-2}{y} dy}$ $\Rightarrow \mu(y) = \dfrac{1}{y^2}$

Multiply the DE by $\mu(y)$:

$\dfrac{1}{y^2}\left(y\,dx + (y^2 - x)\,dy = 0\right)$

$\Rightarrow \dfrac{1}{y}\,dx + \left(\dfrac{y^2 - x}{y^2}\right) dy = 0$ $\Rightarrow \underbrace{\dfrac{1}{y}}_{M}\,dx + \underbrace{\left(1 - \dfrac{x}{y^2}\right)}_{N} dy = 0$

$M_y = \dfrac{-1}{y^2}$, $N_x = \dfrac{-1}{y^2}$ $\Rightarrow M_y = N_x$, Exact

The Soln. is: $\dfrac{\partial f}{\partial x} = M$ | $\dfrac{\partial f}{\partial y} = N$

$\int \dfrac{\partial f}{\partial x} \cdot dx = \int \dfrac{1}{y}\,dx$

$\boxed{f(x,y) = \dfrac{1}{y}x + g(y)}$

$\cancel{-\dfrac{1}{y^2}x} + g'(y) = 1 - \cancel{\dfrac{x}{y^2}}$

$\int g'(y) \overset{dy}{=} \int 1\,dy$

$\boxed{g(y) = y}$

$\therefore \rightarrow \boxed{\dfrac{1}{y}x + y = c}$

Ex) If $\mu(y) = y^k$ is ~~and~~ integrating factor for the non-exact equation

$$xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0$$ find k

Soln. :

Multiply the DE by $\mu(y) = y^k$

$$y^k(xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0)$$

$$\underbrace{xy^{k+1}}_{M}\,dx + \underbrace{(2x^2y^k + 3y^{2+k} - 20y^k)}_{N}\,dy = 0$$ exact

$$\Rightarrow M_y = N_x \Rightarrow \boxed{(k+1)}\underbrace{xy^k} = ④\underbrace{xy^k}$$

$$\Rightarrow k+1 = 4 \quad \Rightarrow \boxed{k = 3}$$

Ex) The ODE $(x^2+y)dx + f(x)dy = 0$ becoms exact by the integrating factor $\mu(x) = x$ find $f(x)$

Soln.:

Multiply by $\mu(x) = x$

$$\underbrace{(x^3+yx)}_{M}dx + \underbrace{x f(x)}_{N}dy = 0$$. exact

$\Rightarrow M_y = N_x \Rightarrow x = N_x$

$\int x\,dx = \int N_x\,dx$

في هذه الحالة g(y) يعتبر c ثابت

$\frac{1}{2}x^2 + g(y) = N \Rightarrow \frac{1}{2}x^2 + \boxed{g(y)} = x f(x)$

$\frac{1}{2}x^2 + c = x f(x)$

$$\boxed{f(x) = \frac{x}{2} + \frac{c}{x}}$$

Ex) The ODE $(x^2+y)\,dx + f(x)\,dy = 0$ becomes exact by the integrating factor $\mu(x) = x$ find $f(x)$

<u>Soln.:</u>

multiply by $\mu(x) = x$

$\Rightarrow (x^3 + xy)\,dx + x\,f(x)\,dy = 0$

$\frac{\partial f}{\partial x} = M$

$\int \frac{\partial f}{\partial x} dx = \int x^3 + xy\,dx$

$f(x,y) = \frac{x^4}{4} + \frac{x^2}{2}y + g(y)$

$\frac{\partial f}{\partial y} = N$

~~$\frac{\partial f}{\partial y}$~~

$\frac{x^2}{2} + g'(y) = x\,f(x)$

Remark: If $\mu(x,y) = x^p y^q$ is an integrating factor for the non-exact DE $M dx + N dy = 0$ then we use the condition

$$\boxed{M_y - N_x = p\frac{N}{x} - q\frac{M}{y}}$$

to find p and q

Ex) Find the integrating factor $\mu(x,y) = x^p y^q$ for $(4xy^2 + 3y)dx + (3x^2y + 2x)dy = 0$

find p and q

Soln.:

$$M_y - N_x = p\cdot\frac{N}{x} - q\cdot\frac{M}{y}$$

$$8xy + 3 - (6xy + 2) = p\left(\frac{3x^2y + 2x}{x}\right) - q\left(\frac{4xy^2 + 3y}{y}\right)$$

$$2xy + 1 = p(3xy + 2) - q(4xy + 3)$$

$$2xy + 1 = 3pxy + 2p - 4qxy \; 3q$$

$$2\underbrace{xy}+1 = \underbrace{(3p - 4q)}\underbrace{xy} + \underbrace{(2p - 3q)}$$

$\Rightarrow 2 = 3p - 4q$ ---①

$\Rightarrow 3\cdot① + -4\cdot②$

$\Rightarrow 1 = 2p - 3q$ — ②

$\Rightarrow 6 = 9p - 12q$

so: $-4 = -8p + 12q$

$\boxed{2 = p}$

$\boxed{q = 1}$

$\therefore \mu(x,y) = x^p y^q \boxed{= x^2 y}$

Homogeneous Equation :

Definition : A function $G(x,y)$ is said to be Homogeneous of order n if $G(tx,ty) = t^n G(x,y)$

Ex)

$G(x,y) = x^2 + 2xy$. Is G homogeneous ?

Soln. $G(tx,ty) = (tx)^2 + 2(tx)(ty)$

$= t^2x^2 + 2t^2xy$

$= t^2(x^2 + 2xy)$

$= t^2 G(x,y)$ homogeneous of order 2

Ex) $G(x,y) = x^2 + y^2 + 2xy^2$. Is G homogeneous?

Soln. $G(tx,ty) = (tx)^2 + (ty)^2 + 2(tx)(ty)^2$

$= t^2x^2 + t^2 \cdot y^2 + 2t^3 xy^2$

$\neq t^n G(x,y)$ not homogeneous

Definition: The DE $M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be homogeneous if $M(x,y)$ and $N(x,y)$ are homogeneous of the same order

Ex) $(x^2+y^2)\,dx + (yx + 2y^2)\,dy = 0$, is DE homogeneous?

Soln. Homogeneous of order 2

Ex) $(x^2+xy)\,dx + (2xy+y^2+\underset{\sim}{5})\,dy = 0$, is DE homogeneou

Soln. Not homogeneous

Ex) $\dfrac{dy}{dx} = \dfrac{4x^3 - xy^2}{2y^3+x^2y}$, is DE homogeneous?

Soln. Homogeneous of order 3

إذا ظهر ثابت قبل ما نحول المعادلة لهذه الصيغة اذا مش homogeneous
اما اذا ظهر بعد ما في مشكلة

To Solve the homogeneous DE $\boxed{\frac{dy}{dx} = f\left(\frac{y}{x}\right)}$:

We use the Substitution $\boxed{V = \frac{y}{x}}$ to transform the homogeneous into separable.

Ex) Solve $x^2 dy - (x^2 + xy + y^2) dx = 0$

Soln. $x^2 dy = (x^2 + xy + y^2) dx$

$$\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2} \longrightarrow \text{homogeneous}$$

لازم دائما بقسم البسط والمقام على اعلى درجة

$$\frac{dy}{dx} = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2$$

نشتق بالنسبة ل (x)

$$\text{take } V = \frac{y}{x} \Rightarrow xV = y$$

$$V + x\frac{dv}{dx} = \frac{dy}{dx}$$

$$V + X\frac{dv}{dx} = 1 + V + V^2$$

$$X\frac{dv}{dx} = 1 + V^2$$

$$\frac{1}{1+V^2} dv = \frac{1}{x} dx \text{ — Separable}$$

لازم تطلع Separable بعد الفرض

$$\int \frac{1}{1+V^2} dv = \int \frac{1}{x} dx$$

$$\tan^{-1} V = \ln|x| + C \Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \ln|x| + c$$

Ex) Solve

$$\frac{dy}{dx} = \frac{2y + \sqrt{x^2 - y^2}}{2x}$$

Soln.

$$\frac{dy}{dx} = \frac{2y}{2x} + \frac{1}{2} \cdot \frac{\sqrt{x^2 - y^2}}{x}$$

$$= \frac{y}{x} + \frac{1}{2}\sqrt{\frac{x^2 - y^2}{x^2}}$$

$$= \frac{y}{x} + \frac{1}{2}\sqrt{\frac{x^2}{x^2} - \frac{y^2}{x^2}}$$

$$\frac{dy}{dx} = \frac{y}{x} + \frac{1}{2}\sqrt{1 - \left(\frac{y}{x}\right)^2}$$ ... homog.

take $V = \frac{y}{x} \Rightarrow xV = y$

$$V + x\frac{dv}{dx} = \frac{dy}{dx}$$

$$\cancel{V} + x\frac{dv}{dx} = \cancel{V} + \frac{1}{2}\sqrt{1 - v^2}$$

$$x\frac{dv}{dx} = \frac{1}{2}\sqrt{1 - v^2}$$

$\Rightarrow \frac{1}{\sqrt{1 - v^2}}\,dv = \frac{1}{2x}\,dx$ ... separable

$\int \frac{1}{\sqrt{1 - v^2}}\,dv = \int \frac{1}{2x}\,dx \Rightarrow \sin^{-1} v = \frac{1}{2}\ln|x| + c$

$$\boxed{y = x\sin\left(\frac{1}{2}\ln|x| + c\right)}$$

نبدل V بـ $\left(\frac{y}{x}\right)$ ونجعل y

موضوع قانون

Ex) solve $x\,dy = y(\ln y - \ln x + 1)\,dx$

Soln.

$$x\,dy = \left(y\ln\left(\frac{y}{x}\right) + y\right)dx$$

$$\frac{dy}{dx} = \frac{y\ln\left(\frac{y}{x}\right)}{x} + \frac{y}{x} \quad \text{.... homog.}$$

take $v = \frac{y}{x} \Rightarrow xv = y \Rightarrow v + x\frac{dv}{dx} = \frac{dy}{dx}$

$$\Rightarrow \cancel{v} + x\frac{dv}{dx} = v\ln(v) \cancel{+v}$$

$$\int \frac{1}{v\ln(v)}\,dv = \int \frac{1}{x}\,dx \quad \text{.... Separable}$$

$$\int \frac{\frac{1}{v}}{\ln(v)}\,dv = \int \frac{1}{x}\,dx$$

$$\ln|\ln(v)| = \ln|x| + c$$

$$e^{\ln\left|\ln\left(\frac{y}{x}\right)\right|} = e^{\ln|x| + c}$$

$$\left|\ln\left(\frac{y}{x}\right)\right| = e^{\ln|x| + c}$$

$$\Rightarrow \boxed{\left|\ln\frac{y}{x}\right| = c|x|}$$

Ex) find the value of k and m so that the DE is homogeneous.

$$(3\sqrt{xy^3} - 3m + 4)\,dx + (x^2 y^k)\,dy = 0$$

Soln.

$$(3\,x^{\frac{1}{2}}y^{\frac{3}{2}} - 3m + 4)\,dx + (x^2 y^K)\,dy = 0$$

homogeneous $\Rightarrow \frac{1}{2} + \frac{3}{2} = 2 + K$

$$\boxed{K=0}$$

and $\Rightarrow$ [redacted] $-3m + 4 = 0 \Rightarrow \boxed{m = \frac{4}{3}}$

Equation of the form $\frac{dy}{dx} = G(ax+by+c)$:

This DE can be reduced to separable equation by using the substitution $V = ax+by+c$

Ex) Solve $\frac{dy}{dx} = 2 + \sqrt{y-2x+3}$

Soln-

$$V = y - 2x + 3$$

$$V + 2x - 3 = y$$

نشتق بالنسبة لـ

$$\frac{dv}{dx} + 2 = \frac{dy}{dx}$$

$$\frac{dv}{dx} \cancel{+2} = \cancel{2} + \sqrt{V + \cancel{2x} \cancel{-3} \cancel{-2x} \cancel{+3}}$$

$$\frac{dv}{dx} = \sqrt{V}$$

$$\Rightarrow \underbrace{\int \frac{1}{\sqrt{V}}\, dV = \int 1\, dx}_{\text{separable}} \qquad \Rightarrow 2\sqrt{V} = x + c$$

$$\Rightarrow \sqrt{V} = \frac{x+c}{2} \qquad \Rightarrow V = \left(\tfrac{1}{2}x + c\right)^2$$

$$y - 2x + 3 = \left(\tfrac{1}{2}x + c\right)^2$$

$$\boxed{y = \left(\tfrac{1}{2}x + c\right)^2 + 2x - 3}$$

[redacted]

c ضرب نص
يضله ثابت c

Ex) Solve $\frac{dy}{dx} = \sin(x-y)$

Soln. $v = x - y \Rightarrow y = x - v$

$$\boxed{\frac{dy}{dx} = 1 - \frac{dv}{dx}}$$

$$1 - \frac{dv}{dx} = \sin v$$

$$\frac{dv}{dx} = 1 - \sin v \Rightarrow \underbrace{\frac{1}{1-\sin v} dv = dx}_{\text{separable}}$$

$$\int \frac{1}{1-\sin v} dv = \int 1 dx$$

$$\Rightarrow \int \frac{1}{1-\sin v} \cdot \frac{1+\sin v}{1+\sin v} dv = x + c$$

$$\Rightarrow \int \frac{1+\sin v}{1-\sin^2 v} dv = x + c$$

$$\Rightarrow \int \frac{1+\sin v}{\cos^2 v} dv = x + c$$

$$\int \left(\frac{1}{\cos^2 v} + \frac{\sin v}{\cos^2 v}\right) dv = x + c$$ ~~$\Rightarrow \int \sec^2 v +$ [illegible]~~

$$\Rightarrow \int (\sec^2 v + \tan v . \sec v) dv = x + c$$

$$\Rightarrow \tan v + \sec v = x + c$$

$$\Rightarrow \boxed{\tan(x-y) + \sec(x-y) = x + c}$$ implicit Soln

Ex) Solve $\frac{dy}{dx} = y - x - 1 + (x - y + 2)$

Soln.

$$\frac{dy}{dx} = \underbrace{y - x}-1 + \frac{1}{\underbrace{x-y}+2}$$

يجب ان يكونوا بنفس الترتيب

$$\frac{dy}{dx} = y - x - 1 + \frac{1}{-(y-x)+2}$$

$$V = y - x \Rightarrow y = V + x$$

$$\boxed{\frac{dy}{dx} = \frac{dv}{dx} + 1}$$

$$\frac{dv}{dx} + 1 = V - 1 + \frac{1}{-V+2}$$

$$\frac{dv}{dx} = V - 2 - \frac{1}{V-2} \Rightarrow \frac{dv}{dx} = \frac{(V-2)^2 - 1}{V-2}$$

$$\frac{dv}{dx} = \frac{V^2 - 4V + 4 - 1}{V-2} \Rightarrow \frac{dv}{dx} = \frac{V^2 - 4V + 3}{V-2}$$

separable $\Rightarrow \frac{1}{2}\int \frac{2(V-2)}{V^2-4V+3}\,dv = \int 1\,dx \Rightarrow \frac{1}{2}\ln|V^2-4V+3| = x + c$

$$\Rightarrow \boxed{\frac{1}{2}\ln|(y-x)^2 - 4(y-x) + 3| = x + c}$$

Ex) Solve $\frac{dy}{dx} = \frac{2y}{x} + \cos\left(\frac{y}{x^2}\right)$

Soln. $v = \frac{y}{x^2}$

$x^2 v = y \Rightarrow \frac{dy}{dx} = 2xv + x^2 \frac{dv}{dx}$

$$\cancel{2xv} + x^2 \frac{dv}{dx} = \cancel{\frac{2x^2 v}{x}} + \cos v$$

$$\int \frac{1}{\cos v}\, dv = \int \frac{1}{x^2}\, dx \Rightarrow \int \sec v\, dv = \int x^{-2}\, dx$$

$$\Rightarrow \ln|\sec v + \tan v| = -\frac{1}{x} + c$$

$$\Rightarrow \boxed{\ln\left|\sec\left(\frac{y}{x^2}\right) + \tan\left(\frac{y}{x^2}\right)\right| = \frac{-1}{x} + c}$$

Ex) The Substitution $V = x^2 \ln y$ transforms the DE

$$x^2 y' + 4x^2 y \ln y = y \sin x$$ into: Linear? in V.

Soln.

$\boxed{V = x^2 \ln y}$ ⇒ [redacted] $\boxed{V' = 2x \ln y + x^2 \cdot \frac{1}{y} \cdot y'}$

↓

$\frac{V}{x} = x \ln y$ نضرب بـ (y) ⇒ $\boxed{yV' = 2xy \ln y + x^2 y'}$

$$\boxed{x^2 y' = yV' - 2xy \ln y}$$

$$yV' - 2xy \ln y + 4x^2 y \ln y = y \sin x$$

[redacted]

$\cdot \frac{1}{y}$ ⇒ $V' - 2x \ln y + 4x^2 \ln y = \sin x$

$$V' - 2\frac{V}{x} + 4V = \sin x$$

$V' + (4 - \frac{2}{x})V = \sin x$ ... $\boxed{\text{Linear in } V}$

Ex) Classify each D.E. as separable, Linear, exact, homogeneous or Bernoulli

① $\frac{dy}{dx} = \frac{x-y}{x}$

⊛ ⇒ Linear in y because: $\frac{dy}{dx} = 1 - \frac{y}{x}$

$\Rightarrow \boxed{\frac{dy}{dx} + \frac{1}{x}y = 1}$

or ⊛ homogeneous: $\frac{dy}{dx} = 1 - \frac{y}{x}$

or ⊛ Exact : $x\,dy = x - y\,dx$

$(y-x)dx + x\,dy = 0$

$M_y = 1$, $N_x = 1$ $\therefore M_y = N_x$ Exact

② $\frac{dy}{dx} = \frac{1}{y-x}$

⊛ ⇒ Linear in x ⇒ $\frac{dx}{dx} = y - x \Rightarrow \boxed{x' + x = y}$

or ⊛ tak $v = y - x$

③ $(x+1)\frac{dy}{dx} = -y+10$

✱ Exact ⟹ $(x+1)\,dy = (-y+10)\,dx$

$\Rightarrow \underbrace{(y-10)}_{M}dx + \underbrace{(x+1)}_{N}dy = 0$

$M_y = 1 \;,\; N_x = 1$

$M_y = N_x \;\therefore$ Exact

✱ Separable because $\frac{1}{-y+10}\,dy = \frac{1}{x+1}\,dx$

✱ Linear in y ⟹ $(x+1)\,y' + y = 10$

$y' + \frac{1}{x+1}\,y = \frac{10}{x+1}$

CH. 3 :

Linear second and Higher order DE

$$a_n(x)\, y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \dots + a_1(x)\, y' + a_0(x)\, y = g(x) \quad \text{①}$$

$(n \geq 2)$

اذا كان g(x) يساوي صفر نسميه Homogeneous

Remark: Ⓐ If $g(x) \neq 0$, then the eq. ① is called non-homogeneous, but If $g(x) = 0$, then eq. ① is called homogeneous

Ex)

$x^2 y'' + 3xy' - 4y = 0$ homogeneous

$x^2 y'' + 2xy' - 5y = e^x$ non-homogeneous

Ⓑ If $a_n(x), a_{n-1}(x), \dots, a_1(x), a_0(x)$ are constants, then $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = g(x)$ is called linear DE with constant coefficients

Ex) $3y'' - 2y' + 5y = 0$ homogeneous with constant coefficients

Theorem : Superposition Principle

If $y_1, y_2, —, y_n$ are solutions for $a_n(x)\, y^{(n)} + — + a_1(x)\, y' + a_0(x)\, y = 0$, then $y = c_1 y_1 + c_2 y_2 + — + c_n y_m$ is a general solution

اي عملية جمع بين ال solutions بتعطيك solution جديد

Definition : If $y_1, y_2, --, y_n$ ~~and~~ are diff. functions of (n-1) order, then the wronskian of $y_1, y_2, —, y_n$ is given as :

$$W[y_1, y_2, —, y_n] = \begin{vmatrix} y_1 & y_2 & \cdots\cdots & y_n \\ y'_1 & y'_2 & \cdots — & y'_n \\ y_1^{(2)} & y_2^{(2)} & \cdots\cdots & y_n^{(2)} \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & — & y_n^{(n-1)} \end{vmatrix}$$

Ex) find اذا اقترانين نشتق مرة واحدة و اذا ٣ اقترانات نشتق مرتين ... الخ

① $W[x, x^2] = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix}$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \cdot d - b \cdot c$$

$= x \cdot 2x - x^2 \cdot 1$

$= 2x^2 - x^2 \boxed{= x^2}$

Ex) find $W[\cosh x, \sinh x] = \begin{vmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{vmatrix}$

$= \cosh^2 x - \sinh^2 x$

$\boxed{= 1}$

Ex) find $W[\cos(2x), \sin(2x)] = \begin{vmatrix} \cos(2x) & \sin(2x) \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$

$2 = \cos^2 2x + 2\sin^2 2x$

$= 2 \cdot 1 \quad \boxed{= 2}$

Ex) find $W[x^2, x^3, \cos x] =$

ارجع اكتبهم

$$\left|\begin{array}{ccc|cc} x^2 & x^3 & \cos x & x^2 & x^3 \\ 2x & 3x^2 & -\sin x & 2x & 3x^2 \\ 2 & 6x & -\cos x & 2 & 6x \end{array}\right.$$

$$-3x^4\cos x - 2x^3\sin x + 12x^2\cos x - (6x^2\cos x - 6x^3\sin x - 2x^4\cos x)$$

$$\Rightarrow -3 \cdot x^4\cos x - 2x^3\sin x + 12x^2\cos x - 6x^2\cos x + 6x^3\sin x + 2x^4\cos x$$

$$\Rightarrow (-3x^4 + 12x^2 - 6x^2 + 2x^4)\cos x + (-2x^3 + 6x^3)\sin x$$

Ex) $W[2, x, x^2, x^3, e^x] =$

$$\begin{vmatrix} 2 & x & x^2 & x^3 & e^x \\ 0 & 1 & 2x & 3x^2 & e^x \\ 0 & 0 & 2 & 6x & e^x \\ 0 & 0 & 0 & 6 & e^x \\ 0 & 0 & 0 & 0 & e^x \end{vmatrix}$$

المصفوفة اللي بكون تحت القطر كله اصفار المحددة هي حاصلة ضرب العناصر على القطر

$\Rightarrow 2 \cdot 1 \cdot 2 \cdot 6 \cdot e^x = 24e^x$

Definition: we say that $y_1, y_2, \dots, y_n$ are Linearly independent on I if $W[y_1, y_2, \dots, y_n] \neq 0$ $\forall x \in I$

otherwise dependent

Ex) Show that $\{2, e^x\}$ are independent on $(-\infty, \infty)$

Soln. : $W[2, e^x] = \begin{vmatrix} 2 & e^x \\ 0 & e^x \end{vmatrix}$

$2e^x - 0 \Rightarrow 2e^x \neq 0$

$\forall x \in (-\infty, \infty)$

$\therefore \{2, e^x\}$ are linearly independent

Ex) is $\{x, x^2\}$ are Independent on $(-8, \infty)$?

Soln. $W[x, x^2] = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2$

$\boxed{= x^2}$

Not Independent since $W[x, x^2](0) = 0^2$

$= 0$

and $0 \in (-8, \infty)$

$\therefore$ So it's dependent

Ex) is $\{0, 2x, 4x\sin x\}$ independent on $(0, \pi]$?

Soln. No this set is dependent

Since $0 \in \{0, 2x, 4x\sin x\}$

Ex) find the value of k such that

$\{2x, 2k-6, e^x\}$ is dependent

Soln. $2k-6=0$

$\boxed{k=3}$

Fundamental set of solution

$\{y_1, y_2, -, y_n\}$ is said to be fundamental set of solution for DE

$$a_n(x)\, y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \cdots + a_1(x)\, y' + a_0(x) y = 0$$

If :

① $y_1, y_2, \rightarrow, y_n$ are solutions for DE

② $W[y_1, y_2, - y_n] \neq 0$ (Independent)

Abel's Theorem

If y_1, y_2 are solution for

$y'' + P(x)\, y' + q(x)\, y = 0$, then

$W[y_1, y_2] = C\, e^{-\int P(x)\, dx}$

Ex) If y_1, y_2 are solution for

$xy'' + 2y' + xy = 0$ find $W[y_1, y_2](x)$.

Soln. $\underbrace{y'' + \frac{2}{x}}_{P(x) = \frac{2}{x}} y' + y = 0$

ال P(x) تكون معامل
ثاني اكبر رتبة اذا
مش موجودة
ثاني اكبر رتبة
اذا P(x) يساوي صفر

$W[y_1, y_2](x) = c\,e^{-\int \frac{2}{x} dx}$

$= c\,e^{-2\ln x} \quad = c\,x^{-2} \boxed{= \frac{c}{x^2}}$

Ex) If $x^2y'' - 2y' + (3+x)y = 0$ if $W[y_1, y_2](2) = 3$, find $W[y_1; y_2](5)$.

Soln. $y'' \underbrace{- \frac{2}{x^2}}_{\hookrightarrow P(x)} y' + \left(\frac{3+x}{x^2}\right) y = 0$

$W[y_1, y_2](x) = c\,e^{-\int -\frac{2}{x^2} dx} \quad \boxed{= c\,e^{-\frac{2}{x}}}$

Since $W[y_1, y_2](2) = 3 \Rightarrow c\,e^{-\frac{2}{2}} = 3$

$c\,e^{-1} = 3 \Rightarrow \boxed{c = 3e} \Rightarrow \therefore W[y_1, y_2](x) = 3e\,e^{-\frac{2}{x}}$.

$\boxed{= 3e^{1-\frac{2}{x}}} \Rightarrow W[y_1, y_2](5) = 3e^{1-\frac{2}{5}} \boxed{= 3e^{\frac{3}{5}}} \rightarrow$ The answer

كيف بتعرف متى تستخدمها؟ لما يطلب انك تستخدمها بعبارة صريحة

أو يقلك جد general solution أو يقلك جد 2nd Soln.

Reduction of order

If y_1 is the solution for DE

$$y'' + P(x)\,y' + q(x)\,y = 0$$, then the second solution is:

$$y_2 = y_1 \cdot \int \frac{e^{-\int P(x)dx}}{(y_1)^2}\,dx$$

and the general solution is $y = c_1 y_1 + c_2 y_2$

Ex) If $y_1 = e^x$ is a solution for the DE

$$xy'' - (x+1)\,y' + y = 0 \;, \quad x>0 .$$ find the second solution and the general solution

Soln.

$$\underbrace{y'' - \frac{(x+1)}{x}}_{P(x)}\, y' + \frac{1}{x}y = 0$$

اذا لا يوجد y' تكون $P(x)=$ze صفر

$$P(x) = -\frac{(x+1)}{x}$$

$$\boxed{P(x) = -1 \; \frac{1}{x}}$$

$$y_2 = e^x \cdot \int \frac{e^{-\int -1 \frac{1}{x}dx}}{e^{2x}}\,dx$$

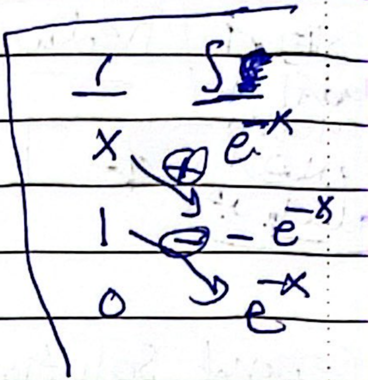
$$\Rightarrow e^x \int \frac{e^{x+\ln x}}{e^{2x}}\,dx \Rightarrow e^x \int \frac{x e^x}{e^{2x}}\,dx \Rightarrow e^x \int x e^{-x}\,dx$$

اجزاء

تكملة الحل ←

$\Rightarrow e^{x}\int x e^{-x}\,dx$

اجزاء



$\Rightarrow e^{x}\left[-x e^{-x} - e^{-x}\right]$

$= -x-1$

$\therefore \boxed{y_2 = -x-1}$ ✓ or $y_2 = x+1$ ✓

the general solution is

$y = c_1y_1 + c_2y_2 = \boxed{c_1e^{x} + c_2(-x-1)}$

Ex) If $y_1 = \cos(\ln x)$ is a solution for DE

$x^2y'' + xy' + y = 0$, find the general solution

Soln. $y'' + \underbrace{\frac{1}{x}}_{} y' + \frac{1}{x^2} y = 0$

$$\boxed{P(x) = \frac{1}{x}}$$

$$y_2 = y_1 \cdot \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$$

$$\Rightarrow y_2 = \cos(\ln x) \cdot \int \frac{e^{-\int \frac{1}{x}dx}}{\cos^2(\ln x)} dx$$

$$= \cos(\ln x) \cdot \int \frac{\sec^2(\ln x)}{x} dx$$

substitution

$$\Rightarrow \cos(\ln x) \cdot \int \frac{\sec^2 v}{\cancel{x}} \cdot \cancel{x} dv$$

$v = \ln x$

$dv = \frac{1}{x} dx$

$x dv = dx$

$$\Rightarrow \cos(\ln x) \cdot \tan \underset{\hookrightarrow \ln x}{v}$$

$$\Rightarrow \cancel{\cos(\ln x)} \cdot \frac{\sin(\ln x)}{\cancel{\cos(\ln x)}} \Rightarrow \boxed{y_2 = \sin(\ln x)}$$

General soln. is : $c_1 \cos(\ln x) + c_2 \sin(\ln x)$

Linear homogeneous second order with constant coefficients

$$ay'' + by' + cy = 0 \quad , \quad a, b, c \text{ are constants and } a \neq 0$$

To solve this DE, we suppose that the solution has the form $y = e^{rx}$, then $y' = re^{rx}$ and $y'' = r^2 e^{rx}$ Put this in DE $\Rightarrow$

$$\boxed{ay'' + by' + cy = 0}$$

$$ar^2 e^{rx} + br e^{rx} + c e^{rx} = 0$$

$$\underbrace{(ar^2 + br + c)}_{=0} \underbrace{e^{rx}}_{\neq 0} = 0$$

$\therefore$ $\boxed{ar^2 + br + c = 0}$ $\rightarrow$ is called the auxiliary eq. for DE.

Then there will be 3 forms of the general soln.:

Case ① If $\overset{\text{المميز}}{\Delta} = b^2 - 4ac > 0$, the we have 2 distinct real roots, say r_1, r_2

$\Rightarrow$ the solutions of DE are $y_1 = e^{r_1 x}$, $y_2 = e^{r_2 x}$ and the general solution is $y = c_1 y_1 + c_2 y_2$

Ex) ① Solve $y'' + 5y' + 6y = 0$

معادلة تفاضلية
من الرتبة الثانية
معاملاتها ثوابت
تساوي صفر (homog.)

Soln. $r^2 + 5r + 6 = 0 \rightarrow$ auxiliary eq.

$(r+2)(r+3) = 0$

$r_1 = -2 \ , \ r_2 = -3$

$\therefore \ y_1 = e^{-2x} \quad , \quad y_2 = e^{-3x}$

$\Rightarrow$ general Soln. is $y = c_1 e^{-2x} + c_2 e^{-3x}$

② Solve $y'' - 4y = 0$

Soln. $r^2 - 4 = 0$ -- . aux. eq.

$(r-2)(r+2)$

$r = 2 \ , \ r = -2$

$\Rightarrow y_1 = e^{2x} \quad , \quad y_2 = e^{-2x}$

the general solution $y = c_1 e^{2x} + c_2 e^{-2x}$

③ Solve $y'' - 5y' = 0$

Soln $r^2 - 5r = 0$

$r(r-5) = 0$

$r_1 = 0$, $r_2 = 5$

$\Rightarrow y_1 = e^{0 \cdot x} = 1$, $y_2 = e^{5x}$

general Solution is $y = c_1 + c_2 e^{5x}$

Case 2,

If $\Delta = b^2 - 4ac = 0$, then we have repeated real root, say: r_1, r_1

$\Rightarrow y_1 = e^{r_1 x}$, $y_2 = x e^{r_1 x}$

لكي لا يكون في تكرار في الحلول ويمكن
اثبات صحتها من خلال الreduction of order

general soln. is: $y = c_1 y_1 + c_2 y_2$

Ex) ① Solve $y'' + 12y' + 36y = 0$

Soln.: $r^2 + 12r + 36 = 0$

$(r+6)(r+6)$

$\Rightarrow r = -6, -6$ مكرر

$\Rightarrow y_1 = e^{-6x}$, $y_2 = x e^{-6x}$

$\therefore$ general soln is: $y = c_1 e^{-6x} + c_2 x e^{-6x}$

② Solve $4y'' + 20y' + 25y = 0$

Soln. $4r^2 + 20r + 25 = 0$

$(2r+5)(2r+5) = 0$ مكرر

$\Rightarrow r = \frac{-5}{2}, \frac{-5}{2}$

$y_1 = e^{-\frac{5}{2}x}$, $y_2 = x e^{-\frac{5}{2}x}$

general Soln. : $y = c_1 e^{-\frac{5}{2}x} + c_2 x e^{-\frac{5}{2}x}$

Case ③ : If $\Delta = b^2 - 4ac < 0$, then we have complex roots say: $\boxed{r = \alpha \pm \beta i}$

(where $\alpha, \beta \in \mathbb{R}$, $i = \sqrt{-1}$)

e^{rx}

$e^{(\alpha+\beta i)x}$

$$\Rightarrow \boxed{y_1 = e^{\alpha x}\cos(\beta x) \ ; \ y_2 = e^{\alpha x}\sin(\beta x)}$$

general soln. is: $y = c_1 y_1 + c_2 y_2$

$\sqrt{-1} = i$
$\sqrt{-4} = 2i$
$\sqrt{-9} = 3i$
$\sqrt{-5} = \sqrt{5}\, i$

Ex) ① Solve $y'' + 2y' + 5y = 0$

Soln. $r^2 + 2r + 5 = 0$

$a = 1$, $b = 2$, $c = 5$

$\Delta = b^2 - 4ac \Rightarrow 4 - 20 = -16 < 0$
complex

$$\Rightarrow r = \frac{-b \pm \sqrt{\Delta}}{2a}$$

$$= \frac{-2 \pm \sqrt{-16}}{2 \cdot 1} = \frac{-2 \pm 4i}{2} \ \boxed{= \underbrace{-1}_{\alpha} \pm \underbrace{2}_{\beta} i}$$

$$\Rightarrow y_1 = e^{\alpha x}\cos(\beta x) \boxed{= e^{-x}\cos(2x)}$$

$$y_2 = e^{\alpha x}\sin(\beta x) \boxed{= e^{-x}\sin(2x)}$$

$\therefore$ general soln. : $\boxed{y = e^{-x}\left(c_1\cos(2x) + c_2\sin(2x)\right)}$

Ex) Solve $y'' + 4y' + 7y = 0$

Soln. $r^2 + 4r + 7 = 0$

$a = 1 \;,\; b = 4 \;,\; c = 7$

$\Delta = b^2 - 4ac = 4^2 - 4 \cdot 1 \cdot 7$

$= -12 < 0$

$\therefore$ complex

$$r = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-4 \pm \sqrt{-12}}{2 \cdot 1} = \frac{-4 \pm 2\sqrt{3}\, i}{2}$$

$r = -2 \pm \sqrt{3}\, i$

$\Leftarrow \boxed{\begin{matrix} \alpha = -2 \\ \beta = \sqrt{3} \end{matrix}}$

$y_1 = e^{\alpha x} \cos(\beta x)$

$y_2 = e^{\alpha x} \sin(\beta x)$

$y_1 = e^{-2x} \cos(\sqrt{3}\, x)$

$y_2 = e^{-2x} \sin(\sqrt{3}\, x)$

General Soln. $\Rightarrow$ $y = C_1 y_1 + C_2 y_2$

$\Rightarrow y = e^{-2x} \left(C_1 \cdot \cos(\sqrt{3}x) + \sin(\sqrt{3}\, x) \cdot C_2 \right)$

Ex) Solve the following DE:

① $y'' + 5y' \mp 6y = 0$

~~Soln.~~ $\cancel{r^2 + 5r + 6 = 0}$

$\cancel{(r+2)(r+3)}$

$\cancel{\boxed{r_1 = -2}}$ $\cancel{\boxed{r_2 = -3}}$

$\cancel{\Rightarrow}$

Soln. $r^2 + 5r - 6 = 0$

$(r-1)(r+6) = 0$

$\boxed{r_1 = 1}$ $\boxed{r_2 = -6}$

$y_1 = e^{x}$ $y_2 = e^{-6x}$

General soln is $\boxed{y = e^{x} c_1 + e^{-6x} c_2}$

② $y'' - 10y' + 25y = 0$

Soln. $r^2 - 10r + 25 = 0$

$(r-5)(r-5)$

$r_1 = 5, 5 \quad , \Delta = 0$

$y_1 = e^{5x} \qquad y_2 = x e^{5x}$

=> General Soln. is $y = c_1 e^{5x} + c_2 x e^{5x}$

③ $y'' - 2y' + 5y = 0$

Soln. $r^2 - 2r + 5 = 0$

$a = 1 , b = -2 , c = 5$

$\Delta = b^2 - 4ac = 4 - 20 \boxed{= -16 < 0}$

$r = \dfrac{-b \pm \sqrt{\Delta}}{2a} = \dfrac{2 \pm \sqrt{-16}}{2} = \dfrac{2 \pm 4i}{2}$

$\boxed{\begin{array}{c} r = 1 \pm 2i \\ \alpha = 1 \\ \beta = 2 \end{array}}$

$y_1 = e^{x} \cos(2x) \;|\; y_2 = e^{x} \sin(2x)$

General Soln => $y = c_1 e^{x} \cos(2x) + c_2 e^{x} \sin(2x)$

$\boxed{y = e^{x} (c_1 \cos(2x) + c_2 \sin(2x))}$

④ $y'' + 9y = 0$

<u>Soln</u> $r^2 + 9 = 0$

$\Rightarrow r^2 = -9 \quad \Rightarrow r = \pm\sqrt{-9}$

$$\boxed{\begin{array}{c} r = \pm 3i \\ \alpha = 0 \\ \beta = 3 \end{array}}$$

$y_1 = e^{0x}\cos(3x) = \cos(3x)$

$y_2 = e^{0x}\sin(3x) = \sin(3x)$

General Soln $\Rightarrow \boxed{y = c_1\cos(3x) + c_2\sin(3x)}$

⑤ $y'' + 6y' + 13y = 0$

$r^2 + 6r + 13 = 0$

$a = 1, b = 6, c = 13$

$\Delta = b^2 - 4ac = 36 - 52 \boxed{= -16 < 0}$

$r = \frac{-b \pm \sqrt{\Delta}}{2a} \Rightarrow \frac{-6 \pm \sqrt{-16}}{2}$ $\boxed{\begin{array}{c} = -3 \pm 2i \\ \alpha = -3 \\ \beta = 2 \end{array}}$

$$\boxed{G.S. \Rightarrow y = c_1 e^{-3x}\cos(2x) + c_2 e^{-3x}\sin(2x)}$$

④ $y'' + 9y = 0$

Soln $r^2 + 9 = 0$

$\Rightarrow r^2 = -9 \quad \Rightarrow r = \pm\sqrt{-9}$

$$\boxed{\begin{array}{c} r = \pm 3i \\ \alpha = 0 \\ \beta = 3 \end{array}}$$

$y_1 = e^{0x}\cos(3x) = \cos(3x)$

$y_2 = e^{0x}\sin(3x) = \sin(3x)$

General Soln $\Rightarrow \boxed{y = c_1\cos(3x) + c_2\sin(3x)}$

⑤ $y'' + 6y' + 13y = 0$

$r^2 + 6r + 13 = 0$

$a = 1, b = 6, c = 13$

$\Delta = b^2 - 4ac = 36 - 52 \boxed{= -16 < 0}$

$r = \frac{-b \pm \sqrt{\Delta}}{2a} \Rightarrow \frac{-6 \pm \sqrt{-16}}{2}$

$$\boxed{\begin{array}{c} = -3 \pm 2i \\ \alpha = -3 \\ \beta = 2 \end{array}}$$

$\boxed{G.S. \Rightarrow y = c_1 e^{-3x}\cos(2x) + c_2 e^{-3x}\sin(2x)}$

Ex) If $y_1 = e^{-5x}$, $y_2 = e^{x}$ are solutions for DE $y'' + (b-3)y' - 4cy = 0$, find b, c.

Soln $\quad y_1 = e^{\overset{r_1}{-5x}} \qquad\qquad y_2 = e^{\overset{r_2}{x}}$

$$r_1 = -5 \qquad\qquad r_2 = 1$$

$$\Rightarrow (r - r_1)(r - r_2) = 0$$

$$(r+5)(r-1) = 0$$

$$r^2 + 4r - 5 = 0$$

$$\underbrace{y'' + 4y' - 5y = 0}, \Rightarrow b - 3 = 4$$

$$\boxed{b = 7}$$

$$\Rightarrow -4c = -5$$

$$\boxed{c = \frac{5}{4}}$$

ممكن يعطيني السؤال على شكل $\boxed{y = c_1 e^{-5x} + c_2 e^{x}}$

⇐ نفس الحل

Ex) find the DE $ay'' + by' + cy = 0$ if it has $y = xe^{3x}$ a solution

Soln.

$r_1 = 3$ $r_2 = 3$ مكرر لأنه مضروب بـ (x)

$$\Rightarrow (r - r_1)(r - r_2) = 0$$

$$(r-3)(r-3) = 0$$

$$r^2 - 6r + 9 = 0$$

$$\boxed{y'' - 6y' + 9y = 0} \quad \#$$

Five Apple

Ex) If $y = e^{-3x}\sin(2x)$ is a solution for DE

$ay'' + (2b-3)y' - cy = 0$, find a, b, c.

Soln.

$$y = e^{\overset{\alpha}{-3}x}\sin(\overset{\beta}{2}x)$$

$$\alpha = -3 \qquad \beta = 2$$

$$r = \alpha \pm \beta i = -3 \pm 2i$$

خطوة ① نقل -3 $\quad r + 3 = \pm 2i$

خطوة ② تربيع $\quad (r+3)^2 = (\pm 2i)^2$

$$r^2 + 6r + 9 = 4i^2$$

$$\boxed{\begin{aligned} i &= \sqrt{-1} \\ i^2 &= -1 \end{aligned}}$$

$$r^2 + 6r + 9 = -4$$

$$r^2 + 6r + 13 = 0$$

$$y'' + 6y' + 13y = 0$$

$\boxed{a=1}$, ~~$b=6$, $C=13$~~

$2b - 3 = 6 \Rightarrow \boxed{b = \frac{9}{2}}$, $-c = 13$

$\boxed{C = -13}$

Ex) Solve the following IVP

$$y'' - 3y' + 2y = 0 \quad , y(0) = 3 \quad , y'(0) = -5$$

Soln.

$$r^2 - 3r + 2 = 0$$

$$(r-2)(r-1) = 0$$

$$\boxed{r=2} \quad \boxed{r=1}$$

$$y_1 = e^{2x} \qquad y_2 = e^{x}$$

$$G.S. \Rightarrow \boxed{y = c_1 e^{2x} + c_2 e^{x}}$$

$$\text{since } y(0)=3 \Rightarrow \boxed{3 = c_1 + c_2} \text{---} ①$$

$$\text{since } y'(0)=-5 \Rightarrow y' = 2c_1 e^{2x} + c_2 e^{x}$$

$$y'(0) = -5 \Rightarrow \boxed{-5 = 2c_1 + c_2} - ②$$

$$3 = c_1 + c_2$$

$$2^{nd} \quad -5 = 2c_1 + c_2$$

$$8 = -c_1 \Rightarrow \boxed{c_1 = -8}$$

$$\therefore \boxed{c_2 = 11}$$

∴ The Solution of IVP is:

$$\boxed{y = -8e^{2x} + 11e^{x}}$$

[redacted] Solve the following IVP:

$$y'' + 2y' + 5y = 0 \quad , \quad y(0) = 3 \quad , \quad y'(0) = -2$$

Soln.

$$r^2 + 2r + 5 = 0$$

$$a = 1 \; , \; b = 2 \; , \; c = 5$$

$$\Delta = b^2 - 4ac \Rightarrow 4 - 20 \boxed{= -16 < 0}$$

$$r = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-2 \pm 4i}{2} \boxed{= -1 \pm 2i}$$

$$\boxed{\begin{array}{c} \alpha = -1 \\ \beta = 2 \end{array}}$$

$$y_1 = e^{-x} \cos(2x)$$

$$y_2 = e^{-x} \sin(2x)$$

G.S. $\Rightarrow$ $y = c_1 e^{-x} \cos(2x) + c_2 e^{-x} \sin(2x)$

Since $y(0) = 3 \Rightarrow \boxed{3 = c_1 + 0} \Rightarrow \boxed{c_1 = 3}$

$$y = 3e^{-x} \cos(2x) + c_2 e^{-x} \sin(2x)$$

$$y' = -6e^{-x} \sin(2x) + (-3e^{-x} \cos(2x)) + 2c_2 e^{-x} \cos(2x) + (-c_2 e^{-x} \sin 2x)$$

$y'(0) = -2 \Rightarrow -2 = -3 + 2c_2$

$$\boxed{c_2 = \frac{1}{2}}$$

∴ The soln. of the IVP is $\Rightarrow$ $\boxed{y = 3e^{-x} \cos(2x) + \frac{1}{2} e^{-x} \sin 2x}$

Linear Homogeneous higher order DE with constant coefficients:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

where $a_0, a_1, \text{---}, a_n$ are constants

Ex) Solve the following DE:

① $y''' - 2y'' - 3y' = 0$

$y' \to r$
$y'' \to r^2$
$y''' \to r^3$
$y^{(4)} \to r^4$
$\vdots$

Soln. $r^3 - 2r^2 - 3r = 0$

$$r(r^2 - 2r - 3) = 0$$

$$r(r-3)(r+1) = 0$$

$$r_1 = 0, \quad r_2 = 3, \quad r_3 = -1$$

$$y_1 = e^{r_1 x} \boxed{= 1}, \quad y_2 = e^{r_2 x} \boxed{= e^{3x}}, \quad \boxed{y_3 = e^{-x}}$$

$$\boxed{y = c_1 + e^{3x} c_2 + e^{-x} c_3}$$

② $y^{(3)} + 3y^{(2)} + 3y^{(1)} + y = 0$

Soln. $\underbrace{r^3}_{} \underbrace{+ 3r^2 + 3r}_{} \underbrace{+1}_{} = 0$

$\Rightarrow \underbrace{r^3+1}_{} \underbrace{+3r^2+3r}_{} = 0$

$\underline{(r+1)}(r^2 - r + 1) + (3r)\underline{(r+1)} = 0$

$(r+1)(r^2 - r + 1 + 3r) = 0$

$\Rightarrow (r+1)(r^2 + 2r + 1) = 0$

$(r+1)(r+1)(r+1) = 0$

$r_1 = -1, \; r_2 = -1, \; r_3 = -1$, so

$y_1 = e^{-x}, \; y_2 = x e^{-x}, \; y_3 = x^2 e^{-x}$

Gn. Soln $\Rightarrow y = c_1 y_1 + c_2 y_2 + c_3 y_3$

$$\boxed{y = e^{-x} c_1 + x e^{-x} c_2 + x^2 e^{-x} c_3}$$

⑧ $y^{(3)} - 8y = 0$

Soln $r^3 - 8 = 0$

$(r-2)(r^2+2r+4) = 0$

$r-2=0$

$\boxed{r_1 = 2}$

$r^2+2r+4=0$

$a=1,\ b=2,\ c=4$

$\Delta = 4-16 \boxed{=-12<0}$ complex

$$r = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-2 \pm 2\sqrt{3}i}{2}$$

$$\boxed{\begin{gathered} r = -1 \pm \sqrt{3}i \\ \alpha = -1 \\ \beta = \sqrt{3} \end{gathered}}$$

$\boxed{y_1 = e^{2x}}$

$\boxed{y_2 = e^{-x}\cos(\sqrt{3}x)}$

$\boxed{y_3 = e^{-x}\sin(\sqrt{3}x)}$

General soln. ⇒ $\boxed{y = e^{2x}c_1 + e^{-x}\cos(\sqrt{3}x)c_2 + e^{-x}\sin(\sqrt{3}x)c_3}$

③ $y^{(3)} + 27y = 0$

Soln. $r^3 + 27 = 0$

$(r+3)(r^2 - 3r + 9) = 0$

$r + 3 = 0$

$\boxed{r_1 = -3}$

$\boxed{y_1 = e^{-3x}}$

$r^2 - 3r + 9 = 0$

$a = 1 , b = -3 , c = 9$

$\Delta = 9 - 36 \boxed{= -27 < 0}$ complex

$r = \frac{-b \pm \sqrt{\Delta}}{2a}$

$r = \frac{3 \pm 3\sqrt{3}i}{2}$

$\boxed{\begin{matrix} \alpha = \frac{3}{2} \\ \beta = \frac{3\sqrt{3}}{2} \end{matrix}}$

$y_2 = e^{\frac{3}{2}x} \cos(\frac{3}{2}x)$

$y_3 = e^{\frac{3}{2}x} \sin(\frac{3}{2}x)$

General Soln. $\Rightarrow$ $\boxed{y = e^{-3x}c_1 + e^{\frac{3}{2}x}\cos(\frac{3\sqrt{3}}{2}x)c_2 + e^{\overset{\sqrt{3}}{\frac{3}{2}}x}\sin(\frac{3}{2}x)c_3}$

Ex) Solve the following DE:

① $y''' - 6y'' + 11y' - 6y = 0$

Soln. $r^3 - 6r^2 + 11r - 6 = 0$

Note that $\boxed{r_1 = 1} \Rightarrow 1^3 - 6(1)^2 + 11(1) - 6 \overset{?}{=} 0$

عوامل الـ 6 $[\pm 1, \pm 2, \pm 3, \pm 6$ ← نجربهم و اللي بيعطينا صفر بناخده

$1 - 6 + 11 - 6 = 0$

$0 = 0$ ✓

نعمل قسمة تركيبية :

Coefficient	r^3	r^2	r	c
	1	-6	11	-6
$\boxed{1}$	↓	1	-5	6
	1	-5	6	0

اول عدد ينزل كما هو دائماً

نضربه بالعدد اللي صفّر $\boxed{=1}$

0 → الباقي

لازم تطلع صفر والا حلك غلط

$\Rightarrow 1 \cdot r^2 + (-5 \cdot r) + 6 = 0 \Rightarrow r^2 - 5r + 6 = 0$

$(r-3)(r-2)$

$\boxed{r_2 = 3}$ $\boxed{r_3 = 2}$

$\Rightarrow y_1 = e^{rx} \boxed{= e^{x}}$

$\boxed{y_2 = e^{3x}}$

$\boxed{y_3 = e^{2x}}$

$\boxed{\therefore G.S. \Rightarrow y = e^{x} c_1 + e^{3x} c_2 + e^{2x} c_3}$

Five Apple

② $y^{(3)} + 3y^{(2)} - 4y = 0$

Soln. $r^3 + 3r^2 - 4 = 0$

عوامل ال (4) ⇒ $\pm 1, \pm 2, \pm 4$

Note that $\boxed{r_1 = 1}$ ⇒ $1+3-4=0=0$ ✱

	r^3	r^2	r	c
	1	3	0	-4
$\boxed{1}$	↓	1	4	4
	1	4	4	0

⇒ $r^2 + 4r + 4 = 0$

$(r+2)(r+2) = 0$

$\boxed{r_2 = -2}$ $\boxed{r_3 = -2}$

$\boxed{y_1 = e^{x}}$

$\boxed{y_2 = e^{-2x}}$

$\boxed{y_3 = xe^{-2x}}$

$\therefore$ G.S. ⇒ $y = e^{x} c_1 + e^{-2x} c_2 + xe^{-2x} c_3$

③ $y^{(5)} - y^{(4)} - 2y^{(3)} = 0$

Soln. $r^5 - r^4 - 2r^3 = 0$

$r^3(r^2 - r - 1) = 0$

$r^3(r-2)(r+1) = 0$

$r_1 = 0$ $r_2 = 0$ $r_3 = 0$ تكرار	$r_4 = 2$	$r_5 = -1$

$\therefore\ y_1 = 1,\ y_2 = x,\ y_3 = x^2,\ y_4 = e^{2x},\ y_5 = e^{-x}$

$$\boxed{G.S. \Rightarrow y = c_1 + xc_2 + x^2c_3 + e^{2x}c_4 + e^{-x}c_5}$$

④ $y^{(6)} - 8y^{(4)} + 16y^{(2)} = 0$

Soln.

$$r^6 - 8r^4 + 16r^2 = 0$$

$$r^2(r^4 - 8r^2 + 16) = 0$$

$$r^2(r^2-4)(r^2-4) = 0$$

$\boxed{\begin{matrix} r_1 = 0 \\ r_2 = 0 \end{matrix}}$ | $r^2 - 4 = 0$, $r = \pm 2$, $\boxed{\begin{matrix} r_3 = 2 \\ r_4 = -2 \end{matrix}}$ | $r^2 - 4 = 0$, $\boxed{\begin{matrix} r_5 = 2 \\ r_6 = -2 \end{matrix}}$

$y_1 = 1$, $y_2 = x$, $y_3 = e^{2x}$, $y_4 = e^{-2x}$, $y_5 = xe^{2x}$,

$y_6 = xe^{-2x}$

$\boxed{G.S. \Rightarrow y = c_1 + xc_2 + e^{2x}c_3 + e^{-2x}c_4 + xe^{2x}c_5 + xe^{-2x}c_6}$

⑤ $y^{(4)} - 81y = 0$

Soln.

$$r^4 - 81 = 0$$

$$(r^2 - 9)\,(r^2 + 9) = 0$$

$r^2 - 9 = 0$

$r = \pm 3$

$\boxed{\begin{array}{l} r_1 = 3 \\ r_2 = -3 \end{array}}$

$r^2 + 9 = 0$

$r^2 = -9$

$r = \pm\sqrt{-9} = \pm 3i$ complex

$\alpha = 0$

$\beta = 3$

$\boxed{y_1 = e^{3x}}$

$\boxed{y_2 = e^{-3x}}$

$\boxed{\begin{array}{l} y_3 = \cos(3x) \\ y_4 = \sin(3x) \end{array}}$

G.S. $\Rightarrow$ ~~$y = 3c_1\, 3x\, c_2$~~ ~~$y = e^{3x} c_1$~~

$$\boxed{y = e^{3x} c_1 + e^{-3x} c_2 + c_3 \cos(3x) + c_4 \sin(3x)}$$

⑥ $y^{(4)} = 0$

<u>Soln</u> $r^4 = 0$

$r = 0, 0, 0, 0$

$y_1 = 1, \; y_2 = x, \; y_3 = x^2, \; y_4 = x^3$

$$\boxed{G.S. \Rightarrow y = c_1 + x c_2 + x^2 c_3 + x^3 c_4}$$

~~H.w~~

~~⑦ $y^{(4)}$~~

H.w:

⑦ $y^{(4)} + 4y^{(2)} + 4y = 0$

⑧ $y''' - 5y'' + 3y' + 9y = 0$

⑨ $y^{(6)} + 5y^{(4)} - 2y^{(3)} - 10y'' + y' + 5y = 0$

Ex) If $y = c_1 e^{2x} + c_2 e^{-5x} + c_3 x e^{-5x}$ is a general soln. for a linear homogeneous DE with constant coefficients, find the DE.

Soln.

$$y = c_1 e^{\overset{r_1}{\boxed{2}}x} + c_2 e^{\overset{r_2}{\boxed{-5}}x} + c_3 x e^{\overset{r_3}{\boxed{-5}}x}$$

$$r_1 = 2, \quad r_2 = -5, \quad r_3 = -5$$

$$(r - r_1)(r - r_2)(r - r_3) = 0$$

$$(r-2)(r+5)(r+5) = 0$$

$$(r-2)(r^2 + 10r + 25) = 0$$

$$r^3 + 8r^2 + 5r - 50 = 0$$

$$y''' + 8y'' + 5y' - 50y = 0$$

Ex) Given $(r+4)(r^2-1)^2(r^2+4)^2=0$ is the auxiliary eq. for linear homogeneous DE with constant coefficients.

① find the order of this DE. $=9$

② find the general solution of this DE.

① order $=9$

②

$r+4=0$

$\boxed{r_1=-4}$

$(r^2-1)^2=0$

$(r^2-1)(r^2-1)=0$

$r=\pm1 \mid r=\pm1$

$r_2=1 \mid r_4=1$

$r_3=-1 \mid r_5=-1$

$(r^2+4)^2=0$

$(r^2+4)(r^2+4)=0$

$r^2+4=0$

$r^2=-4$

$r=\pm\sqrt{-4}$

$r=\pm2i$

$\alpha=0$

$\beta=2$

$r^2+4=0$

$r=\pm\sqrt{-4}$

$r=\pm2i$

$\alpha=0$

$\beta=2$

$y_1=e^{-4x}$, $y_2=e^{x}$ | $y_3=e^{-x}$, $y_4=xe^{x}$ | $y_5=xe^{-x}$

$y_6=\cos(2x)$ | $y_7=\sin(2x)$ | $y_8=x\cos(2x)$ | $y_9=x\sin(2x)$

$$\boxed{G.S \Rightarrow y=e^{-4x}c_1+e^{x}c_2+e^{-x}c_3+xe^{x}c_4+xe^{-x}c_5+c_6\cos(2x)+c_7\sin(2x)+c_8\,x\cos(2x)+c_9\,x\sin(2x)}$$

Cauchy - Euler Equation (homogeneous):

$$ax^2y'' + bxy' + cy = 0 \quad , \quad x > 0$$

رح نتعامل مع ال 2nd order حاليًا .

then the Substitution $\boxed{x = e^t} \longleftrightarrow \boxed{\ln x = t}$
transforms the Cauchy - Euler eq. into:

$$a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + cy = 0$$

Ex) Solve the following DE:

① $x^2y'' - 2xy' - 4y = 0 \quad , \quad x > 0$

Soln this eq. is Cauchy - Euler with

$a = 1 \qquad b = -2 \qquad c = -4$

let $\boxed{x = e^t} \longleftrightarrow \boxed{\ln x = t}$

$$\Rightarrow a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + cy = 0$$

$$= \frac{d^2y}{dt^2} - 3\frac{dy}{dt} - 4y = 0$$

$$\therefore r^2 - 3r - 4 = 0 \Rightarrow (r-4)(r+1) = 0$$

$\boxed{r_1 = 4} \ \boxed{r_2 = -1}$

$y_1(t) = e^{4t} \ | \ y_2(t) = e^{-t}$

$y_1(t) = e^{4t}$, $y_2(t) = e^{-t}$

$(e^t)^4$

$\boxed{y_1(x) = x^4}$ $\boxed{y_2(x) = x^{-1}}$

$$\boxed{\text{G. Soln} \Rightarrow y = x^4 c_1 + x^{-1} c_2}$$

Cauch - Euler Equation (homogeneous):

$$ax^2y'' + bxy' + cy = 0 \quad , \quad x>0$$

then the substitution $\boxed{x=e^t} \longleftrightarrow \boxed{\ln x = t}$ transform the Cachy - Euler Eq. into:

$$a\frac{d^2y}{dt^2} = (b-a)\frac{dy}{dt} + cy = 0$$

Ex) Solve the following DE.:

① $x^2y'' - 2xy' - 4y = 0 , \quad x>0$

Soln. This equ. is cauchy - Euler eq. with

take $\boxed{x=e^t} \longleftrightarrow \boxed{\ln x = t}$

$$\boxed{\begin{array}{l} a=1 \\ b=-2 \\ c=-4 \end{array}}$$

$$\Rightarrow \quad a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + cy = 0$$

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} - 4y = 0$$

Ex) $2xy'' - y' - \frac{2}{x} y = 0$, $x > 0$

Soln.

نضرب المعادلة ب (x) $\Rightarrow 2x^2 y'' - xy' - 2y = 0 \longrightarrow$ Cauchy-Euler

$x = e^t \longleftrightarrow \ln x = t$

$a = 2$
$b = -1$
$c = -2$

$\Rightarrow a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + cy = 0$

$\Rightarrow 2\frac{d^2y}{dt^2} + -3\frac{dy}{dt} - 2y = 0$

$\Rightarrow 2\frac{d^2y}{dt^2} - 3\frac{dy}{dt} - 2y = 0$

$2r^2 - 3r - 2 = 0$

$(2r+1)(r-2)$

$\boxed{r_1 = -\frac{1}{2}}$ $\boxed{r_2 = 2}$

$\boxed{y_1(t) = e^{-\frac{1}{2}t}}$ $\boxed{y_2(t) = e^{2t}}$

$\boxed{y_1(x) = x^{-\frac{1}{2}}}$ $\boxed{y_2(x) = x^2}$

$$\boxed{G.S. \Rightarrow y = x^{-\frac{1}{2}} c_1 + x^2 c_2}$$

Ex) $x^3y'' + x^2y' + 4xy = 0 \quad , \quad x > 0$

Soln, نقسم على x $\Rightarrow x^2y'' + xy' + 4y = 0 \longrightarrow$ Cauchy-Euler

$$\boxed{x = e^t} \longleftrightarrow \boxed{\ln x = t}$$

$a = 1$
$b = 1$
$c = 4$

$$\Rightarrow a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + cy = 0$$

$$\Rightarrow \frac{d^2y}{dt^2} + 4y = 0$$

$$r^2 + 4 = 0$$

$$r = \pm\sqrt{-4} \Rightarrow \boxed{\begin{matrix} r = \pm 2i \\ \alpha = 0 \\ \beta = 2 \end{matrix}}$$

~~$y_1 = \cos$~~

$y_1(t) = \cos(2t)$
$y_2(t) = \sin(2t)$

$y_1(x) = \cos(2\ln x)$
$y_2(x) = \sin(2\ln x)$

G.S. $\Rightarrow y = c_1\cos(2\ln x) + c_2\sin(2\ln x)$

Ex) $x^2 y'' + 3xy' + 3y = 0$

Soln This DE is Cauchy-Euler eq.

$\Rightarrow \quad \boxed{x = e^t} \longleftrightarrow \boxed{\ln x = t}$

$a = 1$
$b = 3$
$c = 3$

$$a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + cy = 0$$

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 3y = 0$$

$$r^2 + 2r + 3 = 0$$

$a = 1 \quad b = 2 \quad c = 3$

$\Delta = b^2 - 4ac$

$\Delta = 4 - 12 \boxed{= -8 < 0}$ complex

$$r = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-2 \pm \sqrt{8}}{2}$$

$$\boxed{\begin{array}{c} r = -1 \pm \sqrt{2}\, i \\ \alpha = -1 \\ \beta = \sqrt{2} \end{array}}$$

$y_1(t) = e^{-t}\cos(\sqrt{2}\,t)$

$y_2(t) = e^{-t}\sin(\sqrt{2}\,t)$

$y_1(x) = x^{-1}\cos(\sqrt{2}\ln x)$, $y_2(x) = x^{-1}\sin(\sqrt{2}\ln x)$

$$\boxed{\text{G.S.} \Rightarrow y = c_1 x^{-1}\cos(\sqrt{2}\ln x) + c_2 x^{-1}\sin(\sqrt{2}\ln x)}$$

[illegible]

Ex) $4x^2y'' + 8xy' + y = 0$, $x > 0$

Soln. This DE is Cauch-Euler eq.

$\boxed{x = e^t} \longleftrightarrow \boxed{\ln x = t}$ | $a = 4$, $b = 8$, $c = 1$

$$\Rightarrow a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + cy = 0$$

$$\Rightarrow 4\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = 0$$

$$4r^2 + 4r + 1 = 0$$

$$(2r+1)(2r+1) = 0$$

$\boxed{r_1 = \frac{-1}{2}}$ $\boxed{r_2 = \frac{-1}{2}}$, so

$\boxed{y_1(t) = e^{-\frac{1}{2}t}}$ $\boxed{y_2(t) = te^{-\frac{1}{2}t}}$

$\boxed{y_1(x) = x^{-\frac{1}{2}}}$, $\boxed{y_2(x) = \ln x \cdot x^{-\frac{1}{2}}}$

$$\boxed{G.S. \Rightarrow y = x^{-\frac{1}{2}}c_1 + c_2 \ln x \cdot x^{-\frac{1}{2}}}$$

$$\boxed{G.S. \Rightarrow y = x^{-\frac{1}{2}}(c_1 + c_2 \ln x)}$$

Ex) If $y = x^3 \ln x$ is a Solution for DE

$a x^2 y'' + b x y' + c y = 0$, find a, b, c.

Soln.

The DE is ~~Cauchy~~ Cauchy – Euler

$\Rightarrow \quad x = e^t \longleftrightarrow \ln x = t$

Since $y = x^3 \ln x$ is a Solution, then

$$y(t) = e^{3t} \cdot t \quad = t\, e^{\overset{r \leftarrow}{\textcircled{3}} t}$$

$r_1 = 3 \quad , \quad r_2 = 3$

$\Rightarrow \quad (r - r_1)(r - r_2) = 0$

$(r - 3)(r - 3) = 0$

$r^2 - 6r + 9 = 0$

~~$y'' - 6y' + 9y = 0$~~

$\Rightarrow \quad \frac{d^2y}{dt^2} - 6\frac{dy}{dt} + 9y = 0$

~~$a = 1$~~ ~~$b = -6$~~ ~~$c = 9$~~

$a\frac{d^2y}{dt^2} + (b - a)\frac{dy}{dt} + cy = 0$

$\boxed{a = 1}$ $\quad b - a = -6 \quad \Rightarrow \boxed{b = -5} \quad \boxed{c = 9}$

$\Rightarrow b - 1 = -6$

Ex) If $y=x^2\cos(2\ln x)$ is a solution for DE

$ax^2y''+bxy'+cy=0$, find a, b, c

Soln.

The DE is Cauchy-Euler $\Rightarrow$ $\boxed{x=e^t \leftrightarrow \ln x=t}$

Since $y=x^2\cos(2\ln x)$ is a Soln.

$$\Rightarrow y(t)=e^{\overset{\alpha\leftarrow}{2t}}\cos(\underset{\beta}{\textcircled{2}}t) \quad \text{Complex}$$

$$r=\alpha\pm\beta i \Rightarrow r=2\pm 2i$$

$$r-2=\pm 2i$$

$$(r-2)^2=(\pm 2i)^2$$

$$\boxed{\begin{matrix}\sqrt{-1}=i\\ (\sqrt{-1})^2=-1\end{matrix}}$$

$$r^2-4r+4=4\cdot -1$$

$$r^2-4r+8=0$$

$$\frac{d^2y}{dt^2}-4\frac{dy}{dt}+8y=0$$

$$a\frac{d^2y}{dt^2}+(b-a)\frac{dy}{dt}+cy=0$$

$\boxed{a=1}$ $\quad b-1=-4 \quad$ $\boxed{c=8}$

$\boxed{b=-3}$

Ex) If $W[2, x, f(x)] = 8x$, find the function $f(x)$

where $f(0) = 2$, $f'(0) = 3$.

Soln.

$$W[2, x, f(x)] = 8x$$

$$\begin{vmatrix} 2 & x & f(x) \\ 0 & 1 & f'(x) \\ 0 & 0 & f''(x) \end{vmatrix} = 8x$$

$$2 \cdot 1 \cdot f''(x) = 8x$$

$$f''(x) = 4x$$

$$\int f''(x)\,dx = \int 4x\,dx$$

$$f'(x) = 2x^2 + C_1$$

Since $f'(0) = 3 \Rightarrow \boxed{3 = C_1}$

$$\int f'(x)\,dx = \int 2x^2 + 3\,dx$$

$$f(x) = \frac{2}{3}x^3 + 3x + C_2$$

Since $f(0) = 2 \Rightarrow \boxed{2 = C_2}$

$$\boxed{f(x) = \frac{2}{3}x^3 + 3x + 2}$$

Ex) If $W[x, y(x)] = x^2 e^x$, find the function $y(x)$ where $y(1) = e^2$

Soln.

$$W[x, y(x)] = x^2 e^x$$

$$\begin{vmatrix} x & y(x) \\ 1 & y'(x) \end{vmatrix} = x^2 e^x$$

$$x y'(x) - y(x) = x^2 e^x$$

$$y'(x) - \underbrace{\frac{1}{x}}_{P(x)} y(x) = x e^x \quad \text{— (linear)}$$

$$\mu(x) = e^{\int P(x) dx} = e^{-\int \frac{1}{x} dx} = x^{-1} \boxed{= \frac{1}{x}}$$

$$y = \frac{1}{\mu(x)} \int \mu(x) g(x) dx + \frac{C}{\mu(x)}$$

$$y = x \int \frac{1}{\cancel{x}} \cdot \cancel{x} e^x dx + Cx$$

$$\boxed{y = x e^x + x C}$$

Since $y(1) = e^2 \Rightarrow e^2 = e + C$

$$\boxed{C = e^2 - e}$$

$$\boxed{y = x e^x + x e^2 - x e}$$

Remark : The DE $ay'' + by' + cy = 0$ can be written as $L[y](x) = 0$ where [redacted])

$L[y](x) = ay'' + by' + cy$ is a linear operator

Ex) If $L[y](x) = 3y'' - 4y' + 5y$. find $L[y](x)$ where $y = x^3 - 4x^2 + 1$

Soln. $y' = 3x^2 - 8x$

$y'' = 6x - 8$

$\therefore L[y](x) = L[x^3 - 4x^2 + 1]$

$= 3(6x-8) - 4(3x^2 - 8x) +$ [redacted]$)$
$5(x^3 - 4x^2 + 1)$

$= 18x - 24 - 12x^2 + 32x + 5x^3 - 20x^2 + 5$

$= 5x^3 - 32x^2 + 50x - 19$

Ex) If $L[y] = y'' - 2y' + 3y$, then find the general soln. for $L[y](x) = 0$

Soln. $L[y](x) = 0$

$y'' - 2y' + 3y = 0$

$r^2 - 2r + 3 = 0$ —— . احل

CH.4 : Non-Homogeneous Linear second and higher order

$$ay'' + by' + cy = g(x)$$

To solve this DE:

① Solve the homog. eq. $ay'' + by' + cy = 0$ to find the homog. Soln. called: $y_h = c_1 y_1 + c_2 y$
or
Complementary Soln.

② Solve the non-homog. to find the Particular Soln. : y_p

③ The general Solution is: $y = y_h + y_p$

To find the Particular Soln. y_p, we use the following methods:

① undetermined Coefficients

② Variation of Parameter

Method ① : Undetermined Coefficients:

$$ay'' + by' + cy = g(x)$$

To use this method, g(x) must be:

- Polynomial , exponential, or (sin, cos)

$$g(x) \begin{cases} \xrightarrow{} \overset{\text{كثير حدود}}{\text{Polynomial}} \\ \xrightarrow{} \sin, \cos \\ \xrightarrow{} e^{\alpha x} \end{cases}$$

مش ضروري يكون كل واحد لحاله، ممكن يكون حاصل ضربهم مثل [redacted] $e^{\alpha x}\sin x$ وغيره .

g(x)	Yp

⊛ Polynomial:

$$5 \longrightarrow (A) \cdot x^{k}$$

$$x+2 \longrightarrow (Ax+b) \cdot x^{k}$$

$$x^2 \longrightarrow (Ax^2+bx+c) \cdot x^{k}$$

K: عدد مرات ظهور $\boxed{\text{الصفر}}$ في جذور المعادلة auxiliary

⊛ Exponential:

m: عدد [redacted] مرات ظهور α في جذور المعادلة

$$3e^{\alpha x} \longrightarrow (Ae^{\alpha x})\, x^{m}$$

$$xe^{\alpha x} \longrightarrow ((Ax+b)e^{\alpha x})\, x^{m}$$

$$x^2e^{\alpha x} \longrightarrow ((Ax^2+bx+c)e^{\alpha x})\, x^{m}$$

⊛ Sin or cos :   ما ينطبق على ال sin
ينطبق على ال cos

$2\sin(\beta x) \longrightarrow (A\cos(\beta x) + B\sin(\beta x))\,x^n$

$x\sin(\beta x) \longrightarrow ((Ax+B)\cos(\beta x) + (Cx+D)\sin(\beta x))\,x$

$xe^{\alpha x}\sin(\beta x) \longrightarrow ((Ax+B)e^{\alpha x}\cos(\beta x) + (Cx+D)e^{\alpha x}\sin(\beta x)$
x

n : عدد ظهور العدد $r = \alpha + \beta i$ في جذور المعادلة

في الحالة الأولى والثانية نلاحظ عدم وجود exponential اذا α تكون صفر

Ex) find the general soln. for the DE

$$y'' - 2y' - 3y = \underbrace{4x - 5 - 6xe^{2x}}_{g(x)}$$

Soln. ① homogeneous : $y'' - 2y' - 3y = 0$

$y_1 = e^{3x}$ $y_2 = e^{-x}$

$\boxed{y_h = c_1 e^{3x} + c_2 e^{-x}}$

$r^2 - 2r - 3 = 0$

$(r-3)(r+1) = 0$

$\boxed{r_1 = 3}$ $\boxed{r_2 = -1}$

② Non-homog. :

$$g(x) = \underbrace{4x-5}_{Poly} + \underbrace{6xe^{2x}}_{expo.}$$

دائماً هنا (+) لأنه الإشارات تحدد على حساب A و B

$$y_p = (Ax+B)x^K + ((Cx+D)e^{2x})\,x^m$$

K : عدد مرات ظهور الصفر في الجذر والمقصود من الجذور r_1 و r_2 $\Rightarrow K = 0$

m : عدد مرات ظهور 2 في الجذور $\Rightarrow m = 0$

$\Rightarrow$ $\boxed{y_p = (Ax+B) + (Cx+D)e^{2x}}$ → The form of particular sol

تكملة الحل →

To find the constant A, B, C, D, then:

$$y'_p = A + Ce^{2x} + 2(Cx+D)e^{2x}$$

$$y''_p = 2Ce^{2x} + 2Ce^{2x} + 4(Cx+D)e^{2x}$$

نعوض في المعادلة التفاضلية ⇒ Put y_p, y'_p, y''_p in the DE:

$$y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$$

$$4Ce^{2x} + 4(Cx+D)e^{2x} - 2(A + Ce^{2x} + 2(Cx+D)e^{2x}) - 3(Ax + B + (Cx+D)e^{2x}) = 4x - 5 + 6xe^{2x}$$

$$\Rightarrow \underbrace{(-2A-3B)}_{\text{ثابت}} + \underbrace{(2C-3D)}_{\text{معامل } e^{2x}} e^{2x} \underbrace{-3A}_{\text{معامل } x}x \underbrace{-3C}_{\text{معامل } xe^{2x}}xe^{2x} = \underbrace{4}_{\text{معامل } x}x \underbrace{-5}_{\text{ثابت}} + \underbrace{6}_{\text{معامل } xe^{2x}}xe$$

ثابت ⟶ $-2A - 3B = -5 \Rightarrow \boxed{B = \frac{7}{9}}$

معامل e^{2x} → $2C - 3D = 0 \Rightarrow \boxed{D = \frac{-4}{3}}$

معامل x $\quad -3A = 4 \Rightarrow \boxed{A = \frac{-4}{3}}$

معامل xe^{2x} $\quad -3C = 6 \Rightarrow \boxed{C = -2}$

∴ The Particular Solution is:

$$y_p = \frac{-4}{3}x + \frac{7}{9} + \left(-2x - \frac{4}{3}\right)e^{2x}$$

③ The G. Soln. ⇒ $y = y_h + y_p \Rightarrow y = c_1e^{3x} + c_2e^{-x} - \frac{4}{3}x + \frac{7}{9} + \left(-2x - \frac{4}{3}\right)e$

Ex) find the General solution for the DE:

$$y'' + 4y = \cos(2x)$$

Soln.

① homog. $y'' + 4y = 0$

$r^2 + 4 = 0$

$r^2 = -4 \Rightarrow \boxed{r = \pm 2i}$

$y_1 = \cos(2x)$

$y_2 = \sin(2x)$

$$\boxed{y_h = C_1\cos(2x) + C_2\sin(2x)}$$

② Non-homog. :

$$g(x) = \cos(2x)$$

$$y_p = [A\cos(2x) + B\sin(2x)]\,x^n$$

$r = \alpha + \beta i$ n عدد مرات ظهور

$= 0 + 2i$

ظهور r=2i مرة واحدة $\leftarrow \boxed{= 2i} \Rightarrow \boxed{n = 1}$

$\therefore n = 1$

$$\Rightarrow \boxed{y_p = Ax\cos(2x) + Bx\sin(2x)} \rightarrow \text{form for particular soln.}$$

نشتق ونعوض في المعادلة التفاضلية

~~y_p~~ To find the constants: A, B then:

$$y'_p = A\cos 2x - 2Ax\sin 2x + B\sin 2x + 2Bx\cos$$

$$y''_p = -2A\sin 2x - 2A\sin 2x - 4Ax\cos 2x + 2B\cos 2x + 2B\cos(2x) - 4Bx\sin 2x$$

$$y''_p = -4A\sin 2x - 4Ax\cos 2x + 4B\cos 2x - 4Bx\sin 2x$$

→ Put y_p, y_p', y_p'' in DE:

$$y'' + 4y = \cos 2x$$

$$-4A\sin 2x - 4A\cancel{x\cos 2x} + 4B\cos 2x - \cancel{4Bx\sin 2x} +$$
$$4\cancel{Ax\cos 2x} + \cancel{4Bx\sin 2x} = \cos 2x$$

$\therefore \quad -4A\sin 2x + 4B\cos 2x = \cos 2x$

معامل الsin $\Rightarrow -4A = 0 \Rightarrow \boxed{A = 0}$

معامل ال $\Rightarrow 4B = 1 \Rightarrow \boxed{B = \frac{1}{4}}$

∴ Particular soln is:

$$y_p = \frac{1}{4}x\sin(2x)$$

⑧ Gn Soln.: $y = y_h + y_p$

$$\boxed{y = c_1\cos 2x + c_2\sin 2x + \frac{1}{4}x\sin(2x)}$$

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Scanned with CamScanner

Ex) find a form of particular soln. for the following DE (without calculating the coefficients):

① $y'' - 5y' + 6y = 2\sin 4x + 5e^{2x}$

Soln. homog. ⇒ $y'' - 5y' + 6y = 0$

$r^2 - 5r + 6 = 0$

$(r-2)(r-3)$

$\boxed{r_1 = 2}$ $\boxed{r_2 = 3}$

$y_1 = e^{2x}$

$y_2 = e^{3x}$

$$\boxed{y_h = e^{2x} c_1 + e^{3x} c_2}$$

② non-homog. ⇒

$g(x) = 2\sin 4x + 5e^{2x}$

$y_p = (A\cos 4x + B\sin 4x)x^n + (Ce^{2x})x^m$

$n = 0$ | $m = 1$

$r = \alpha + \beta i$

$= 0 + 4i$

$\boxed{= 4i}$ ✗

$$\boxed{y_p = A\cos 4x + B\sin 4x + Cxe^{2x}}$$

↳ form of particular soln

② $y'' + 2y' = e^{-x} - y$

<u>soln.</u>

نتقل جميع المرافق ال y على [redacted] جهة والباقي على الجهة الاخرة → $y'' + 2y' + y = e^{-x}$

① homog. $y'' + 2y' + y = 0$

$$y_1 = e^{-x} \qquad \qquad r^2 + 2r + 1 = 0$$

$$(r+1)(r+1) = 0$$

$$y_2 = x e^{-x} \qquad \qquad \boxed{r_1 = -1} \quad \boxed{r_2 = -1}$$

② Non-homog $g(x) = e^{-x}$

$$y_p = (Ae^{-x})\, x^m$$

$$\boxed{m = 2}$$

$$\boxed{y_p = Ax^2 e^{-x}}$$

Five Apple

③ $y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}$

Soln.

① homog. $y'' - 6y' + 9y = 0$

$r^2 - 6r + 9 = 0$

$(r-3)(r-3) = 0$

$\boxed{r_1 = 3}$ $\boxed{r_2 = 3}$

$y_1 = e^{3x}$

$y_2 = xe^{3x}$

$$y_h = e^{3x}c_1 + xe^{3x}c_2$$

② Non-homog. $\Rightarrow g(x) = 6x^2 + 2 - 12e^{3x}$

$$y_p = (Ax^2 + Bx + c)x^k + (De^{3x})x^m$$

$\boxed{k=0}$ $\boxed{m=2}$

$$\boxed{\therefore y_p = Ax^2 + Bx + c + Dx^2e^{3x}}$$

④ $y^{(4)} - 2y^{(3)} + y^{(2)} = 1 + e^x$

Soln. homog. $y^{(4)} - 2y^{(3)} + y^{(2)} = 0$

$r^4 - 2r^3 + r^2 = 0$

$r^2(r^2 - 2r + 1) = 0$

$r^2(r-1)(r-1) = 0$

$\boxed{r_1 = 0}$ $\boxed{r_2 = 0}$ $\boxed{r_3 = 1}$ $\boxed{r_4 = 1}$

$y_1 = 1$

$y_2 = x$

$y_3 = e^x$

$y_4 = xe^x$

② non-homog. $\Rightarrow g(x) = \underbrace{1} + \underbrace{e^x}$

$$y_p = (A)x^k + (Be^x)x^m$$

$\boxed{k=2}$ $\boxed{m=2}$

$\boxed{y_p = Ax^2 + \overset{\beta}{B}x^2e^x}$ → form of P. Soln.

⑤ $y''' + 4y'' + 3y' = x^2 \cos x - 3x e^{-3x}$

Soln.

① homog. $\Rightarrow$ $y''' + 4y'' + 3y' = 0$

$y_1 = 1$

$y_2 = e^{-3x}$

$y_3 = e^{-x}$

$r^3 + 4r^2 + 3r = 0$

$r(r^2 + 4r + 3) = 0$

$r(r+3)(r+1) = 0$

$\boxed{r_1 = 0}$ $\boxed{r_2 = -3}$ $\boxed{r_3 = -1}$

② Non-homog. $\Rightarrow$ $g(x) = \underbrace{x^2 \cos x}_{} - \underbrace{3x e^{-3x}}_{}$

$y_p = \left((Ax^2 + Bx + C)\cos x + (Dx^2 + Hx + G)\sin x\right)x^n +$

$\left((wx + B)e^{-3x}\right)x^m$

اكتبها قد تكون علامات

n: عدد مرات تكرار العدد $\alpha + \beta i$

$= 0 \pm 1i$ $\boxed{= i}$ $\Rightarrow$ $\boxed{n = 0}$

m: عدد مرات ظهور $\alpha = -3$ في الجذور

$\boxed{m = 1}$

$\Rightarrow$ $\boxed{y_p = (Ax^2 + Bx + C)\cos x + (Dx^2 + Hx + G)\sin x + (wx^2 + Vx)\bar{e}^{3}}$

⑥ $y^{(3)} - 2y^{(2)} + y^{(1)} = x + 2e^{x} - \cos(3x)$

<u>Soln.</u> ① homog. $y^{(3)} - 2y^{(2)} + y^{(1)} = 0$

$r^3 - 2r^2 + r = 0$

$r(r^2 - 2r + 1) = 0$

$r(r-1)(r-1) = 0$

$\boxed{r_1 = 0}$ $\boxed{r_2 = 1}$ $\boxed{r_3 = 1}$

$y_1 = 1$

$y_2 = e^{x}$

$y_3 = x e^{x}$

② non-homog. $\Rightarrow g(x) = \underbrace{x}_{} + \underbrace{2e^{x}}_{} - \underbrace{\cos(3x)}_{}$

$y_p = (Ax + B)x^{k} + (Ce^{x})x^{m} + (D\sin(3x) + \overset{H}{\cancel{E}}\cos(3x))x$

لا تنسى الجملة $\boxed{k=1}$ // $\boxed{m=2}$ // $\boxed{n=0}$

$r = \alpha + \beta i$

$= 0 + 3i$

$\boxed{= 3i}$

$$\Rightarrow \boxed{y_p = Ax^2 + Bx + Cx^2 e^{x} + D\sin(3x) + H\cos(3x)}$$

⑦ $y'' + 4y = \cos^2 x$

Soln. ① homog. $\Rightarrow$ $y'' + 4y = 0$

$r^2 + 4 = 0$

$r^2 = -4$

$\boxed{r = \pm 2i}$

$y_1 = \cos 2x$

$y_2 = \sin 2x$

② non-homog $\Rightarrow$ $g(x) = \cos^2 x$

$g(x) = \frac{1}{2} + \frac{1}{2}\cos 2x$

$y_p = (A)x^k + (B\cos 2x + C\sin 2x)x^n$

$\boxed{k=0}$ $\boxed{n=1}$

$\alpha + \beta i$

$0 + 2i$

$\boxed{=2i}$

$\Rightarrow \boxed{y_p = A + Bx\cos 2x + Cx\sin 2x}$

⑧ $y'' + y = \sin x \cos 2x$

Soln. ① homog. $y'' + y = 0$

$$r^2 + 1 = 0 \quad\Big|\quad y_1 = \cos x$$

$$r^2 = -1 \quad\Big|\quad y_2 = \sin x$$

$$\boxed{r = \pm 1i}$$

② non-homog. ⇒ $g(x) = \sin x \cos 2x$

$$= \frac{1}{2}\left[\sin(-x) + \sin(3x)\right]$$

$$= \underbrace{-\frac{1}{2}\sin x}_{} + \underbrace{\frac{1}{2}\sin 3x}_{}$$

$$y_p = (A\cos x + B\sin x)x^{n_1} + (C\cos 3x + D\sin 3x)x^{n_2}$$

// $\boxed{n_1 = 1}$ // $\boxed{n_2 = 0}$

$$\boxed{y_p = Ax\cos x + Bx\sin x + C\cos 3x + D\sin 3x}$$

⑨ $x^2y'' + 3xy' - 8y = (\ln x)^3 - \ln x, \quad x > 0$

Soln. ① Euler eq. $\Rightarrow \boxed{x = e^t} \leftrightarrow \boxed{\ln x = t}$

$a = 1$
$b = 3$
$c = -8$

$$\Rightarrow a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + cy = g(t)$$

$$\Rightarrow \boxed{\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 8y = t^3 - t}$$

~~② non-homog. ⇒ g(x) = (Lnx)³ − Lnx~~

homog. $r^2 + 2r - 8 = 0$

$(r+4)(r-2) = 0$

$\boxed{r_1 = -4} \quad \boxed{r_2 = 2}$

$y_1(t) = e^{-4t}$
$y_2(t) = e^{2t}$

② non-homog: $g(t) = t^3 - t$

$$y_p(t) = (At^3 + Bt^2 + Ct + D)\,t^K$$

K=0 ، عدد مرات ظهور صفر في الجذور

$$\boxed{y_p(t) = At^3 + Bt^2 + Ct + D}$$

$$\boxed{y_p(x) = A(\ln x)^3 + B(\ln x)^2 + C\ln x + D}$$

→

كمل حل $\Leftarrow$ $A=\frac{-1}{8}$, $B=\frac{-3}{32}$,

$C=\frac{-1}{64}$, $D=\frac{-7}{250}$

انا بدي تحل على ال(t) بتعوض في معادلة ال (t) — $\frac{d^2y}{dt^2}$ بعدما [illegible] تشتق $y_p(t)$ و ازا

بدك على (x) بتعوض في معادلة ال (x) [illegible]
لكن الاسهل ال (t)

⑩ $x^2 y'' - xy' + y = 2x \quad , x > 0$

Soln. Euler eq. $x = e^t \leftrightarrow \ln x = t$

$a = 1$
$b = -1$
$c = 1$

$$\Rightarrow a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + cy = g(t)$$

$$\Rightarrow \frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = 2e^t$$

① homog. $\Rightarrow$ $r^2 - 2r + 1 = 0$

$(r-1)(r-1) = 0$

$\boxed{r_1=1}$ $\boxed{r_2=1}$

$y_1 = e^t$

$y_2 = xe^t$

② non-homog. $\Rightarrow$ $g(t) = 2e^t$

$y_{p(t)} = (Ae^t) t^m$

$\boxed{m=2}$

$y_{p(t)} = At^2 e^t$

$$\Rightarrow \boxed{y_p(x) = A(\ln x)^2 x}$$

⑪ $x^2 y'' + x y' + y = \ln x + \sin(\ln x)$, $x > 0$

Soln. Euler eq. $\Rightarrow x = e^t \leftrightarrow \ln x = t$

$$\Rightarrow a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + cy = \overset{g(t)}{\text{[illegible]}} \quad \Bigg| \begin{array}{l} a=1 \\ b=1 \\ c=1 \end{array}$$

$$\Rightarrow \frac{d^2y}{dt^2} + y = t + \sin(t)$$

① homog. $\Rightarrow r^2 + 1 = 0$

$r^2 = -1$

$\boxed{r = \pm 1i}$ $\quad\Bigg|\quad y_1 = \cos(t)$, $y_2 = \sin(t)$

② Non-homog. $\Rightarrow g(t) = \underbrace{t} + \underbrace{\sin(t)}$

~~$y_p(t) =$ [illegible] $+ (At+B)x^1$~~

$$y_p(t) = (At+B)t^K + \cancel{A}(C\cos(t) + D\sin(t))\cancel{x}^n$$

// $\boxed{K=0}$ // $\boxed{n=1}$

$$y_p(t) = At + B + Ct\cos(t) + Dt\sin(t)$$

$$\boxed{y_p(x) = A\ln x + B + C\ln x\cos(\ln x) + D\ln x\sin(\ln x)}$$

Ex) find the Cauchy-Euler eq. of second order general soln. :

$$y(x) = c_1 x + c_2 x \ln x + x(\ln x)^2$$

Soln. : Euler eq. $\Rightarrow$ $x = e^t \leftrightarrow \ln x = t$

$$\Rightarrow \text{g. soln.}: y(t) = \underbrace{c_1 e^t + c_2 e^t t}_{y_h} + \underbrace{e^t t^2}_{y_p}$$

[redacted]

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انتبه النهائي Second o

$\Rightarrow$ g.s: $y = y_h + y_p$

① homog. $\Rightarrow y_h = c_1 e^t + c_2 t e^t$

$$\Rightarrow \boxed{r_1 = 1} \quad \boxed{r_2 = 1}$$

$$(r - r_1)(r - r_2) = 0$$
$$(r - 1)(r - 1) = 0$$
$$r^2 - 2r + 1 = 0$$

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = 0 \Leftrightarrow a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + cy$$

$$\Rightarrow \boxed{a = 1} \quad b - a = -2$$
$$b - 1 = -2 \quad \boxed{c = 1}$$
$$\boxed{b = -1}$$

② Non-homog. ! $y_p(t) = t^2 e^t$ is a Particular Soln. [redacted]
for $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = g(t)$

$\longrightarrow$

$$y_p' = 2te^t + t^2 e^t$$

$$y_p'' = 2e^t + 2te^t + 2te^t + t^2 e^t$$

$$= 2e^t + 4te^t + t^2 e^t$$

Put y_p, y_p', y_p'' in DE

$$\Rightarrow 2e^t + 4te^t + t^2e^t - 2(2te^t + t^2e^t) + t^2e^t = g(t)$$

$$\Rightarrow 2e^t + \cancel{4te^t} + \cancel{t^2e^t} - \cancel{4te^t} - \cancel{2t^2e^t} + \cancel{t^2e^t} = g(t)$$

$$2e^t = g(t)$$

$$g(t) = 2e^t$$

$$\boxed{g(x) = 2x}$$

∴ Euler eq:

$$ax^2y'' + bxy' + cy = g(x)$$

$$x^2y'' + xy' + y = 2x$$

H.w) Solve the IVP

$$y''' + 2y'' - 9y' - 18y = -18x^2 - 18x + 22$$

if $y(0) = -2$

$y'(0) = -8$

$y''(0) = -12$

H.w) Find the form of particular solution:

① $y''' + y'' \overset{-}{\cancel{+}} 5y' + 3y = e^{-x} + \sin x$

② $y^{(4)} - 2y''' + y'' = x^3 e^{-x} \cos x + x^2$

Method ② : Variation of Parameters

$$\underbrace{a}y'' + by' + cy = \underbrace{f(x)}$$

Steps :

① Find y_1, y_2, for homog. DE: $ay'' + by' + cy = 0$

$$\Rightarrow y_h = c_1 y_1 + c_2 y_2$$

② Find $W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

③ Find

[illegible]

$$\left.\begin{aligned} v_1 &= -\int \frac{g(x)\cdot y_2}{W}.dx \\ v_2 &= \int \frac{g(x)\cdot y_1}{W}.dx \end{aligned}\right\} \rightarrow \boxed{g(x) = \frac{f(x)}{a}}$$

④ The Particular soln. is:

$$y_p = y_1 \cdot v_1 + y_2 \cdot v_2$$

⑤ G. soln. $\Rightarrow y = y_h + y_p$

Ex) Find the G. Soln. for : $y'' - 2y' + y = \frac{e^x}{x^2+1}$

Soln- Variation of Parameters:

① homog. $r^2 - 2r + 1 = 0$

$(r-1)(r-1) = 0$

$\boxed{r_1 = 1}$ $\boxed{r_2 = 1}$

$y_1 = e^x$

$y_2 = xe^x$

$y_h = e^x c_1 + xe^x c_2$

② $W[y_1, y_2] = W[e^x, xe^x] = \begin{array}{c|cc|} f & e^x & xe^x \\ f' & e^x & xe^x + e^x \end{array}$

$= \cancel{xe^{2x}} + e^{2x} - \cancel{xe^{2x}} = e^{2x}$

$\boxed{W[e^x, xe^x] = e^{2x}}$

③ (*) $g(x) = \frac{\frac{e^x}{x^2+1}}{1} = \frac{e^x}{x^2+1}$

$V_1 = -\int \frac{\frac{\cancel{e^x}}{x^2+1} \cdot x\cancel{e^x}}{\cancel{e^{2x}}} dx \Rightarrow V_1 = -\int \frac{x}{x^2+1} \cdot dx$

$V_1 = -\frac{1}{2}\int \frac{2x}{x^2+1} dx$

$\boxed{V_1 = \frac{-1}{2} \ln(x^2+1)}$

$V_2 = \int \frac{\frac{\cancel{e^x}}{x^2+1} \cdot \cancel{e^x}}{\cancel{e^{2x}}} dx \Rightarrow V_2 = \int \frac{1}{x^2+1} dx$

$\boxed{V_2 = \tan^{-1}(x)}$

→

~~y_p~~

④ Particular Soln.

$$y_p = y_1 \cdot V_1 + y_2 \cdot V_2$$

$$= e^x \cdot \frac{-1}{2} Ln(x^2+1) + x e^x \tan^{-1} x$$

$$\boxed{= -\frac{e^x}{2} Ln(x^2+1) + x e^x \tan^{-1} x}$$

⑤ Gn. Soln.

$$y = y_h + y_p$$

$$\boxed{y = c_1 e^x + c_2 x e^x + x e^x \tan^{-1} x - \frac{e^x}{2} Ln(x^2+1)}$$

Ex) Find the general soln. for DE:

$$y'' + 4y = \tan(2x)$$

Soln. Variation of Parameters

① homog. $y'' + 4y = 0$

$y_1 = \cos(2x)$ | $r^2 + 4 = 0 \Rightarrow r^2 = -4$

$y_2 = \sin(2x)$ | $\boxed{r = \pm 2i}$

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

② $W[y_1, y_2] = \begin{array}{c} f \\ f' \end{array} \begin{vmatrix} \cos(2x) & \sin(2x) \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$

$= 2\cos^2 2x + 2\sin^2 2x = 2(\cos^2 2x + \sin^2 2x)$

$= 2 \cdot 1 \boxed{= 2}$

③ $g(x) = \frac{\tan(2x)}{1} \boxed{= \tan(2x)}$

~~$v_1 = -\int \frac{\tan(2x) \cdot \cos(2x)}{2} dx \Rightarrow v_1 = -\frac{1}{2} \int (\sin 2x) dx$~~

~~$v_2 = \int \frac{\tan(2x) \cdot \sin(2x)}{2} dx \Rightarrow v_2$~~ ~~$\boxed{v_1 = \frac{1}{4} \cos 2x}$~~

→

$V_1 = -\int \frac{\tan(2x) \cdot \sin 2x}{2} \, dx$

$V_1 = -\int \frac{\sin(2x) \cdot \sin(2x)}{2\cos(2x)} \, dx$

$V_1 = \frac{-1}{2} \int \frac{\sin^2(2x)}{\cos(2x)} \, dx$

$V_1 = -\frac{1}{2} \int \frac{1 - \cos^2(2x)}{\cos(2x)} \, dx$

$V_1 = -\frac{1}{2} \int \sec(2x) - \cos(2x) \, dx$

$V_1 = \frac{-1}{2} \left[\frac{1}{2} \ln|\sec(2x) + \tan(2x)| - \frac{\sin 2x}{2} \right]$

$\boxed{V_1 = -\frac{1}{4} \ln|\sec(2x) + \tan(2x)| + \frac{1}{4} \sin(2x)}$

$V_2 = \int \frac{\tan(2x) \cos(2x)}{2} \, dx \Rightarrow V_2 = \frac{1}{2} \int \sin(2x) \, d$

$\boxed{V_2 = -\frac{1}{4} \cos(2x)}$

④ Particular soln. $\Rightarrow$ ~~y_p~~ $y_p = y_1 V_1 + y_2 V_2$

$y_p = \cos(2x) \cdot \left[-\frac{1}{4} \ln|\sec(2x) + \tan(2x)| + \frac{1}{4} \sin(x) \right]$

$- \frac{1}{4} \sin(2x) \cos(2x)$

$\Rightarrow y_p = \frac{-1}{4} \cos(2x) \cdot \ln|\sec(2x) + \tan(2x)|$

⑤ G.Soln. $\Rightarrow y = y_h + y_p$

Ex) let $L[y](x) = y'' + P(x)y' + q(x)$ and

$\{x^{-6}, x^{-1}\}$ be a fundamental soln set to the DE:

$L[y](x) = 0$

If $y_p = x^{-6}v_1 + x^{-1}v_2$ is a particular soln for the DE $L[y](x) = \frac{e^x}{x^2}$. find v_2

Soln.

⊛ $y_1 = x^{-6}$ $y_2 = x^{-1}$

⊛ $W[y_1, y_2] = \begin{vmatrix} x^{-6} & x^{-1} \\ -6x^{-7} & -x^{-2} \end{vmatrix}$

$-x^{-8} + 6x^{-8} \boxed{= 5x^{-8}}$

$g(x) = \frac{\frac{e^x}{x^2}}{1} \boxed{= \frac{e^x}{x^2}}$

$v_2 = \int \frac{g(x) \cdot y_1}{W} \cdot dx \Rightarrow \int \frac{\frac{e^x}{\cancel{x^2}} \cdot \cancel{x^{-6}}}{5\cancel{x^{-8}}} dx \Rightarrow \frac{1}{5}\int e^x dx$

$\boxed{v_2 = \frac{1}{5} e^x}$

Theorem:

If y_1 is a solution for DE:
$L[y](x) = ay'' + by' + cy = f_1(x)$ and y_2 is a solution
for DE: $L[y](x) = ay'' + by' + cy = f_2(x)$, then $y = c_1 y_1 + c_2 y_2$
is a soln. for DE $L[y](x) = ay'' + by' + cy = c_1 f_1(x) + c_2 f_2(x)$

Ex) If $y_1 = \frac{3}{2}x - \frac{9}{4}$ is a soln. for DE: $y'' + 3y' + 2y = \boxed{3x}$ f_1

and $y_2 = \frac{e^{3x}}{2}$ is a solution for DE: $y'' + 3y' + 2y = \boxed{10e^{3x}}$ f_2

find the soln. for DE: $y'' + 3y' + 2y = \boxed{-9x + 20e^{3x}}$ $f(x)$

Soln $f(x) = c_1 f_1(x) + c_2 f_2(x)$

$$\underbrace{-9}x + \underbrace{20}e^{3x} = \underbrace{c_1 3}x + \underbrace{c_2 10}e^{3x}$$

=بالمقارنة $\Rightarrow -9 = 3c_1 \Rightarrow \boxed{c_1 = -3}$

$\cancel{20} = \cancel{10} c_2 \Rightarrow \boxed{c_2 = 2}$

$\therefore$ The soln. is $y = c_1 y_1 + c_2 y_2$

$$= -3\left(\frac{3}{2}x - \frac{9}{4}\right) + 2\left(\frac{e^{3x}}{2}\right)$$

$$\boxed{y = -\frac{9}{2}x + \frac{27}{4} + e^{3x}}$$

H.w) If y_1 is soln. for DE: $y'' + x^2 y' + y = 5x + 3$
and y_2 is a soln. for DE: $y'' + x^2 y' + y = x^2 - 2x$
find the soln. for DE: $y'' + x^2 y' + y = 2x^2 + 11x + 9$

Ex) find the G. Soln. for DE $y'' - 2y' - 5 = 0$

Soln. $y'' - 2y' = 5$ non-homog.

use undetermined coefficient:

⊛ homog. $y'' - 2y' = 0$

$r^2 - 2r = 0$ | $y_1 = 1$

$r(r-2) = 0$ | $y_2 = e^{2x}$

$\boxed{r_1 = 0}$ $\boxed{r_2 = 2}$

$y_h = c_1 + e^{2x} c_2$

⊛ non-homog. ⇒ $g(x) = 5$

$y_p = (A)x^k$ ⇒ $y_p = Ax$

$\boxed{k=1}$

⊛ $y_p' = A$ $y_p'' = 0$

Put y_p, y_p', y_p'' in DE: $y'' - 2y' = 5$

$0 - 2A = 5$

$\boxed{A = \frac{-5}{2}}$

$\boxed{\therefore y_p(x) = \frac{-5}{2} x}$

Ex) If $-x$, $\frac{x^2}{2}-x$ are solutions for the DE:

$x^3 y''' + f(x) y'' - y = g(x)$. find f(x) and g(x)

Soln.

$y_1 = -x$ is a solution $\Rightarrow y_1' = -1$

$y_1'' = 0$

$y_1''' = 0$

Put y_1, y_1', y_1'', y_1''' in DE:

$0 + 0 + x = g(x) \Rightarrow \boxed{g(x) = x}$

$y_2 = \frac{x^2}{2} - x$ is a soln. $\Rightarrow y_2' = x - 1$

$y_2'' = 1$

$y_2''' = 0$

Put y_2, y_2', y_2'', y_2''' in DE

$\Rightarrow \quad 0 + f(x) \cdot 1 - \left(\frac{x^2}{2} - x\right) = x$

$\Rightarrow \boxed{f(x) = \frac{x^2}{2}}$

Ex) use the undetermined coefficients to find a form of particular soln. for:

$$y^{(3)} + 6y'' + 9y' + 4y = e^{-x} + 3^{x}$$

Soln. ① homog. $\Rightarrow r^3 + 6r^2 + 9r + 4 = 0$

$\boxed{r_1 = -1} \Rightarrow (-1)^3 + 6(-1)^2 + 9(-1) + 4 = -1 + 6 - 9 + 4 = 0$ ✓

	r^3	r^2	r	ثابت
	1	6	9	4
$\boxed{-1}$	↓	-1	-5	-4
	1	5	4	0

$$1 \cdot r^2 + 5 \cdot r + 4 = 0$$

$r^2 + 5r + 4 = 0$

$(r+4)(r+1) = 0$

$\boxed{r_2 = -4}$ $\boxed{r_3 = -1}$

$y_1 = e^{-x}$

$y_2 = e^{-4x}$

$y_3 = x e^{-x}$

Non-homog. $\Rightarrow g(x) = e^{-x} + 3^{x}$

$\Rightarrow \boxed{g(x) = e^{-x} + e^{(\ln 3)x}}$

$3^{x} = e^{\ln(3^{x})}$

$= e^{x \ln 3}$

$= e^{(\ln 3)x}$

~~$y_p = (Ae^{-x})x^m$~~ ~~$\boxed{m=2}$~~

$$y_p = (Ae^{-x})x^{m_1} + (Be^{(\ln 3)x})x^{m_2}$$

$$\boxed{m=2} \qquad \boxed{m_2=0}$$

$$\boxed{y_p = Ax^2e^{-x} + Be^{(\ln 3)x}}$$

$x^2y'' - 2xy' + 2y = x^3e^{-x}$ $\frac{x^3}{e^x}$ $(\ln a)$

$x = e^t$

$a = 1$ $b = -2$ $c = 2$ $\ln x = t$

$\cancel{r^2 - 2r}$

$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2 = \boxed{\frac{t^3}{e^t}}$

$r^2 - 3r + 2 = 0$

$(r-2)(r-1)$

$\boxed{r_1 = 2}$ $\boxed{r_2 = 1}$

$y_1 = e^{2t}$

$y_2 = e^t$

$y_h(t) = e^{2t}c_1 + e^tc_2$

$\boxed{y_h(x) = x^2c_1 + xc_2}$

$W[e^{2t}, e^t] = \begin{vmatrix} e^{2t} & e^t \\ 2e^{2t} & e^t \end{vmatrix}$

$e^{3t} - 2e^{3t}$ $\boxed{= -e^{3t}}$

$\boxed{g(t) = \frac{t^3}{e^t}}$ $v_1 = -\int \frac{\frac{t^3}{e^t} \cdot e^t}{-e^{3t}} \, dt \Rightarrow + \int t^3 e^{3t} d$

$\frac{L}{}$ S

t^3

$3t^2$

$6t$

6

0

CH.5 : Laplace Transforms

Defn: let f(t) be a function defined on [0,∞] then the laplace transform for f(t) is given as: $\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\cdot e^{-st}\,dt = F(s)$

Provided that the integral converges

Ex: find $\mathcal{L}\{3\}$.

soln. $f(t) = 3$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{3\} = \int_0^\infty 3e^{-st}\,dt$$

$$= \lim_{h\to\infty}\int_0^h 3e^{-st}\,dt$$

$$= \lim_{h\to\infty} \frac{3e^{-st}}{-s}\Big|_0^h$$

$$= \frac{-3}{s}\lim_{h\to\infty} e^{-sh} - \overset{1}{e^{0}}$$

[redacted] $= \frac{-3}{s}\left[e^{-\infty} - 1\right]$

$e^{-\infty} = \frac{1}{\infty} = 0$

$$\boxed{= \frac{3}{s}}$$

Remark:

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
a	$\frac{a}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$

$n = 1, 2, 3, \ldots$

Remark:

$$\mathcal{L}\{\alpha f(t) \pm \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} \pm \beta \mathcal{L}\{g(t)\}$$

α و β ثوابت

Ex: Evaluate

① $\mathcal{L}\{5\} = \frac{5}{s}$

② $\mathcal{L}\{\cos(3t)\} = \frac{s}{s^2+9}$

③ $\mathcal{L}\{2\sin(5t) - 4e^{2t} + 10t^5 - 2\} = 2\mathcal{L}\{\sin(5t)\} - 4\mathcal{L}\{e^{2t}\} + 10\mathcal{L}\{t^5\} - \mathcal{L}\{2$

$= 2 \cdot \frac{5}{s^2+25} - 4 \cdot \frac{1}{s-2} + 10 \cdot \frac{5!}{s^6} - \frac{2}{s}$

$= \frac{10}{s^2+25} - \frac{4}{s-2} + \frac{1200}{s^6} - \frac{2}{s}$

④ $\mathcal{L}\{\sinh(3t)\} = \mathcal{L}\left\{\frac{e^{3t}-e^{-3t}}{2}\right\}$

$= \frac{1}{2}\,\mathcal{L}\{e^{3t}-e^{-3t}\}$

$= \frac{1}{2}\left(\mathcal{L}\{e^{3t}\} - \mathcal{L}\{e^{-3t}\}\right)$

$= \frac{1}{2}\left(\frac{1}{s-3} - \frac{1}{s+3}\right)$

$\sinh\alpha = \frac{e^{\alpha}-e^{-\alpha}}{2}$

$\cosh\alpha = \frac{e^{\alpha}+e^{-\alpha}}{2}$

⑤ $\mathcal{L}\{\sin(2t+2\pi)\} = \mathcal{L}\{\sin(2t)\}$

$= \frac{2}{s^2+4}$

اي زاوية نضيفلها
او اطرح منها 2π
بترجع بنفس
المكان اذا
بنشيلها

Remark: ① $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$

② $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

③ $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

④ $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha-\beta) + \sin(\alpha+\beta)]$

⑤ $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$

⑥ $\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$

⑦ $\cos(\frac{\pi}{2}-\alpha) = \sin\alpha$ ⑧ $\sin(\frac{\pi}{2}-\alpha) = \cos\alpha$

⑨ $\sin^2\alpha = \frac{1}{2} - \frac{1}{2}\cos(2\alpha)$ ⑩ $\cos^2\alpha = \frac{1}{2} + \frac{1}{2}\cos(2\alpha)$

Ex) Evaluate

① $\mathcal{L}\{\sin(3t-\frac{\pi}{6})\} = \mathcal{L}\{\sin(3t)\cos\frac{\pi}{6} - \sin\frac{\pi}{6}\cos(3t)\}$

$\mathcal{L}\{\frac{\sqrt{3}}{2}\sin(3t) - \frac{1}{2}\cos(3t)\} = \frac{\sqrt{3}}{2}\mathcal{L}\{\sin(3t)\} - \frac{1}{2}\mathcal{L}\{\cos(3$

$$\boxed{= \frac{\sqrt{3}}{2}\cdot\frac{3}{s^2+9} - \frac{1}{2}\cdot\frac{s}{s^2+9}}$$

② $\mathcal{L}\{\sin^2(3t)\} = \mathcal{L}\{\frac{1}{2} - \frac{1}{2}\cos(6t)\}$

$= \mathcal{L}\{\frac{1}{2}\} - \frac{1}{2}\mathcal{L}\{\cos(6t)\}$

$= \frac{1}{2s} - \frac{1}{2}\cdot\frac{s}{s^2+36}$

③ $\mathcal{L}\{\cos(2t)\cdot\cos(5t)\} = \mathcal{L}\{\frac{1}{2}[\cos(-3t)+\cos(7t)]\}$

$= \frac{1}{2}[\mathcal{L}\{\cos(3t)\} + \mathcal{L}\{\cos(7t)\}]$

$= \frac{1}{2}[\frac{s}{s^2+9} + \frac{s}{s^2+49}]$

H.w. ④ $\mathcal{L}\{\sin(3t)\cdot\sin(5t)\}$

~~⑤ $\int_0^\infty e^{-st+10}\,dt = \int_0^\infty e^{10t}\cdot e^{-st}\,dt$~~

⑤ $\int_0^\infty e^{-st+10t}\,dt = \int_0^\infty e^{10t}\cdot e^{-st}\,dt$ $\boxed{\mathcal{L}\{f(t)\}=\int_0^\infty f(t)\,e^{-st}\,d}$

$= \mathcal{L}\{e^{10t}\} = \frac{1}{s-10}$

ازا كان مكان
ال (S) ع (2) مثال
بنحط $\frac{1}{2-10}$

⑥ $\int_0^\infty e^{\overset{s}{-2t}}\cos(3t)\,dt$

$= \mathcal{L}\{\cos(3t)\}_{s=2} = \frac{s}{s^2+9}\Big|_{s=2} \boxed{= \frac{2}{13}}$

Remark: Shifting Property

$\mathcal{L}\{e^{at}\cdot f(t)\} = \mathcal{L}\{f(t)\}\Big|_{s\to s-a} = F(s-a)$

Ex) Evaluate

① $\mathcal{L}\{e^{2t}\cos(5t)\} = \mathcal{L}\{\cos(5t)\}\Big|_{s\to s-2}$

$= \frac{s}{s^2+25}\Big|_{s\to s-2}$ $\boxed{= \frac{s-2}{(s-2)^2+25}}$

② $\mathcal{L}\{t^3 e^{-4t}\} = \mathcal{L}\{t^3\}\Big|_{s\to s+4} = \frac{3!}{s^4}\Big|_{s\to s+4}$

$\boxed{= \frac{6}{(s+4)^4}}$

③ $\mathcal{L}\left\{\frac{\sin 3t}{e^{2t}}\right\} = \mathcal{L}\left[e^{-2t}\sin 3t\right\} = \mathcal{L}\{\sin(3t)\}\Big|_{s\to s+2}$

$= \frac{3}{s^2+9}\Big|_{s\to s+2}$ $\boxed{= \frac{3}{(s+2)^2+9}}$

Remark:

n عدد صحيح موجب

$$\mathcal{L}\{t^n \cdot f(t)\} = (-1)^n \frac{d^n}{ds^n}\left[\mathcal{L}\{f(t)\}\right]$$

Ex) Evaluate

① $\mathcal{L}\{t^{\overset{=n}{2}}\cos(2t)\} = (-1)^2 \frac{d^2}{ds^2}\left[\frac{s}{s^2+4}\right]$

$= \frac{d}{ds}\left[\frac{(s^2+4)\cdot 1 - s(2s)}{(s^2+4)^2}\right] = \frac{d}{ds}\left[\frac{s^2+4-2s^2}{(s^2+4)^2}\right]$

$= \frac{d}{ds}\left[\frac{4-s^2}{(s^2+4)^2}\right] \Rightarrow \frac{d^2}{ds^2} = \frac{-2s\left((s^2+4)^2\right) - (4-s^2)\,2(s^2+4)\cdot 2s}{(s^2+4)^4}$

$= \frac{-2s\cdot(s^2+4)^{\cancel{2}} - 4s(4-s^2)\cancel{(s^2+4)}}{(s^2+4)^{\cancel{4}\,3}} = \frac{-2s^3-8s-16s+4s}{(s^2+4)^3}$

$$\boxed{= \frac{2s^3-24s}{(s^2+4)^3}}$$

② $\mathcal{L}\{t^{20}\cdot e^{3t}\} = \mathcal{L}\{t^{20}\}\Big|_{s\to s-3}$

$= \frac{20!}{s^{21}}\Big|_{s\to s-3}$ $\quad \boxed{= \frac{20!}{(s-3)^{21}}}$

③ $\mathcal{L}\{t e^{4t}\cos(3t)\} = \mathcal{L}\{t\cos(3t)\}\Big|_{s\to s-4}$

$= \left[(-1)^1 \frac{d}{ds}\left[\mathcal{L}\{\cos(3t)\}\right]\right]\Big|_{s\to s-4}$

$= \left[-\frac{d}{ds}\left[\frac{s}{s^2+9}\right]\right]\Big|_{s\to s-4}$

$= \left(\frac{-(s^2+9-2s^2)}{(s^2+9)^2}\right)\Big|_{s\to s-4}$

$= \frac{s^2-9}{(s^2+9)^2}\Big|_{s\to s-4}$ $\quad \boxed{= \frac{(s-4)^2-9}{((s-4)^2+9)^2}}$

Inverse Laplace Transform:

Defn. If $\mathcal{L}\{f(t)\} = F(s)$, then the inverse Laplace is given as: $\mathcal{L}^{-1}\{F(s)\} = f(t)$

$F(s) = \mathcal{L}\{f(t)\}$	$\mathcal{L}^{-1}\{F(s)\} = f(t)$
$\frac{a}{s}$	a
$\frac{1}{s-a}$	e^{at}
$\frac{1}{s^2+a^2}$	$\frac{\sin at}{a}$
$\frac{s}{s^2+a^2}$	$\cos(at)$
$\frac{1}{s^n}$	$\frac{t^{n-1}}{(n-1)!}$

$n = 1, 2, 3, \ldots$

Remark:

① $\mathcal{L}^{-1}\{F(s-a)\} = e^{at}\mathcal{L}^{-1}\{F(s)\}$

② $\mathcal{L}^{-1}\{F(s)\} = \frac{-1}{t}\mathcal{L}^{-1}\{F'(s)\}$

③ $\mathcal{L}^{-1}\{\alpha F(s) \pm \beta G(s)\} = \alpha \mathcal{L}^{-1}\{F(s)\} \pm \beta \mathcal{L}^{-1}\{G(s)\}$

Ex) Evaluate

① $\mathcal{L}^{-1}\left\{\frac{5}{s}\right\} = 5$

② $\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{\sin(3t)}{3}$

③ $\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\} = \cos(4t)$

④ $\mathcal{L}^{-1}\left\{\frac{3}{s^9} - \frac{2s}{s^2+4} + \frac{2}{s} + \frac{4}{s-6}\right\} =$

$= 3\mathcal{L}^{-1}\left\{\frac{1}{s^9}\right\} - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{s-6}\right\}$

$= 3\frac{t^8}{8!} - 2\cos(2t) + 2.1 + 4e^{6t}$

⑤ $\mathcal{L}^{-1}\left\{(s+7)^{-5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+7)^5}\right\} = e^{-7t} \cdot \mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\}$

$= e^{-7t} \cdot \frac{t^4}{4!}$

⑥ $\mathcal{L}^{-1}\left\{\frac{7}{(s-3)^2+25}\right\} = 7\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2+25}\right\} = 7e^{3t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+25}\right\}$

$= 7e^{3t} \cdot \frac{\sin 5t}{5}$

⑦ $\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+16}\right\} = e^{2t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\} = e^{2t}\cos(4t)$

⑧ $\mathcal{L}^{-1}\left\{\frac{s}{(s+3)^2+25}\right\} =$

$$\mathcal{L}^{-1}\left\{\frac{s+3-3}{(s+3)^2+25}\right\} = \mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+25}\right\} - 3\,\mathcal{L}^{-1}\left\{\frac{1}{(s+3)^2+25}\right\}$$

$$= e^{-3t}\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+25}\right\} - 3e^{-3t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+25}\right\}$$

$$= e^{-3t}\cos(5t) - 3e^{-3t}\,\frac{\sin(5t)}{5}$$

⑨ $\mathcal{L}^{-1}\left\{\frac{2}{s^2-4s-5}\right\} \overset{\text{كسور جزئية}}{\Longrightarrow} \frac{2}{s^2-4s-5} = \frac{2}{(s-5)(s+1)}$

$$= \frac{A}{s-5} + \frac{B}{s+1}$$

ازا كانت العبارة
التربيعية في المقام
تحلل إذا ع كسور جزئي
اما إذا كانت لاتحلل
ع اكمال مربع

$$\Rightarrow \boxed{2 = A(s+1) + B(s-5)}$$

$$\text{if } \boxed{s=-1} \Rightarrow 2 = -6B \Rightarrow \boxed{B = \frac{-1}{3}}$$

$$\text{if } \boxed{s=5} \Rightarrow 2 = 6A \Rightarrow \boxed{A = \frac{1}{3}}$$

$$\mathcal{L}^{-1}\left\{\frac{\frac{1}{3}}{s-5} + \frac{\frac{-1}{3}}{s+1}\right\} = \frac{1}{3}\,\mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\} - \frac{1}{3}\,\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= \frac{1}{3}e^{5t} - \frac{1}{3}e^{-t}$$

⑩ $\mathcal{L}^{-1}\left\{\frac{3}{s^2+4s+3}\right\} =$ كسور جزئية ~~$3\mathcal{L}^{-1}\left\{\frac{1}{(s+3)(s+1)}\right\}$~~

~~$\Rightarrow \frac{3}{s^2+4s+3} = \frac{A}{(s+1)} + \frac{B}{(s+3)}$~~

طريقة ثانية
يفضل الطريقة
الأولى

$$\frac{3}{s^2+4s+3} = 3 \cdot \frac{1}{(s+1)(s+3)}$$

$$= \frac{3}{-2} \cdot \frac{\boxed{(s+1)-(s+3)}^{=-2}}{(s+1)(s+3)}$$

$$= \frac{-3}{2} \cdot \left[\frac{\cancel{s+1}}{\cancel{(s+1)}(s+3)} - \frac{\cancel{s+3}}{(s+1)\cancel{(s+3)}}\right]$$

$$= \frac{-3}{2}\left[\frac{1}{s+3} - \frac{1}{s+1}\right]$$

$$\Rightarrow \frac{-3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+3} - \frac{1}{s+1}\right\} = \frac{-3}{2}\left[e^{-3t} - e^{-t}\right]$$

⑪ $\mathcal{L}^{-1}\left\{\frac{2}{s^2+4s+13}\right\} =$ اكمال المربع

يجب أن يكون معامل (s^2) ال $= 1$ → مهم!!!

$s^2 + 4s + 13$

$\left(\frac{s \text{ معامل}}{2}\right)^2 = \left(\frac{4}{2}\right)^2 = 4$

$\Rightarrow s^2 + 4s + 13 + 4 - 4$

$\Rightarrow \boxed{s^2 + 4s + 4} \; \boxed{+13 - 4}$

$\Rightarrow (s+2)^2 + 9$

$2)\ 2\,\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+9}\right\}$

$= 2e^{-2t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}$

$\boxed{= 2e^{-2t} \cdot \frac{\sin(3t)}{3}}$

⑫ $\mathcal{L}^{-1}\left\{\frac{2}{(s^2+9)(s^2+4)}\right\} = 2\,\mathcal{L}^{-1}\left\{\frac{1}{(s^2+9)(s^2+4)}\right\}$

يفضل استخدام الكسور الجزئية لأنه بعض المدرسين ما بحبوا هذه الطريقة !!

$= \frac{2}{5}\,\mathcal{L}^{-1}\left\{\frac{\overset{=5}{(s^2+9)-(s^2+4)}}{(s^2+9)(s^2+4)}\right\}$

$= \frac{2}{5}\,\mathcal{L}^{-1}\left\{\frac{\cancel{(s^2+9)}}{\cancel{(s^2+9)}(s^2+4)} - \frac{\cancel{s^2+4}}{(s^2+9)\cancel{(s^2+4)}}\right\}$

$= \frac{2}{5}\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+4} - \frac{1}{s^2+9}\right\}$

$= \frac{2}{5}\left[\frac{\sin(2t)}{2} - \frac{\sin(3t)}{3}\right]$

⑬ $\mathcal{L}^{-1}\left\{\ln\left(\frac{s+4}{s+5}\right)\right\}$

$\mathcal{L}^{-1}\{F(s)\}=\frac{-1}{t}\mathcal{L}^{-1}\{F'$

$= \mathcal{L}^{-1}\{\ln(s+4)-\ln(s+5)\}$

$= \frac{-1}{t}\mathcal{L}^{-1}\{[\ln(s+4)-\ln(s+5)]'\}$

$= \frac{-1}{t}\mathcal{L}^{-1}\left\{\frac{1}{s+4}-\frac{1}{s+5}\right\}$

$= \frac{-1}{t}\left[e^{-4t}-e^{-5t}\right]$

H.W: Evaluate

① $\mathcal{L}^{-1}\left\{\frac{s+8}{(s+3)^2+81}\right\}$

② $\mathcal{L}^{-1}\left\{\frac{s}{s^2-4s+5}\right\}$

③ $\mathcal{L}^{-1}\left\{\frac{2}{s^2+3s-4}\right\}$

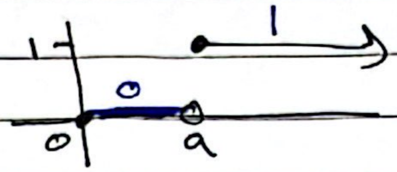
④ $\mathcal{L}^{-1}\left\{\frac{3}{(s^2+1)(s^2+5)}\right\}$

⑤ $\mathcal{L}^{-1}\left\{\ln\left(\frac{s^2}{s^2+16}\right)\right\}$

⑥ ~~$\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right.$~~ $\mathcal{L}^{-1}\left\{\frac{2}{s^3+3s^2+2s}\right\}$

* unit-Step function:

$$u(t-a) = \begin{cases} 0 , & 0 \le t < a \\ 1 , & t \ge a \end{cases}$$



Remark:

① $\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$

② $\mathcal{L}\{u(t-a) f(t)\} = e^{-as} \mathcal{L}\{f(t+a)\}$

③ $\mathcal{L}^{-1}\{e^{-as} \cdot F(s)\} = u(t-a) \cdot \mathcal{L}^{-1}\{F(s)\}\Big|_{t \to t-a}$

Ex) Evaluate

① $\mathcal{L}\{u(t-2)\} = \frac{e^{-2s}}{s}$

بسبب ال 100 ما يوجد في الفترة مرة

∴ مضروب بـ (1)

② $\mathcal{L}\{(u(t-3))^{100}\} = \mathcal{L}\{u(t-3)\} = \frac{e^{-3s}}{s}$

③ $\mathcal{L}\{u(t-5)(t^2+t+2)\} =$

$= e^{-5s}\mathcal{L}\{(t+5)^2+(t+5)+2\}$

$= e^{-5s}\mathcal{L}\{t^2+10t+25+t+5+2\}$

$= e^{-5s}\mathcal{L}\{t^2+11t+32\}$

$= e^{-5s}\left(\frac{2!}{s^3}+\frac{11}{s^2}+\frac{32}{s}\right)$

④ $\mathcal{L}\{u(t-\frac{\pi}{3})\cos(2t)\}$

$= e^{-\frac{\pi}{3}s}\mathcal{L}\{\cos(2(t+\frac{\pi}{3}))\} \Rightarrow e^{-\frac{\pi}{3}s}\mathcal{L}\{\cos(2t+\frac{2\pi}{3})$

$= e^{-\frac{\pi}{3}s}\mathcal{L}\{\cos(2t)\overset{-\frac{1}{2}}{\cos(\frac{2\pi}{3})} - \sin(2t)\overset{\frac{\sqrt{3}}{2}}{\sin(\frac{2\pi}{3})}\}$

$= e^{-\frac{\pi}{3}s}\left(\frac{-1}{2}\mathcal{L}\{\cos(2t)\} - \frac{\sqrt{3}}{2}\mathcal{L}\{\sin(2t)\}\right.$

$= e^{-\frac{\pi}{3}s}\left(-\frac{1}{2}\cdot\frac{s}{s^2+4} - \frac{\sqrt{3}}{\cancel{2}}\cdot\frac{\cancel{2}}{s^2+4}\right)$

Five Apple

⑤ $\mathcal{L}\{u(t-2)\,e^{3t}\} = \mathcal{L}\{u(t-2)\}_{s \to s-3}$

$= \frac{e^{-2s}}{s}\Big|_{s \to s-3}$ $\boxed{= \frac{e^{-2(s-3)}}{s-3}}$

Ex) If $g(t) = \begin{cases} t, & t < 2 \\ 3, & 2 < t < 4 \\ e^{-3t}, & 4 < t \end{cases}$

اول فترة يجب
أن تكون ال (t)
اقل من عدد
(يجب أن تكون الفترات
على ~~الترتيب~~ شكل متصاعد)

Find $\mathcal{L}\{g(t)\}$.

Soln.

$g(t) = t + u(t-2)\cdot(3-t) + u(t-4)\cdot(e^{-3t}-3)$

$\mathcal{L}\{g(t)\} = \mathcal{L}\{t\} + \mathcal{L}\{u(t-2)\,(3-t)\} + \mathcal{L}\{u(t-4)\cdot(e^{-3t}-3)\}$

$= \frac{1}{s^2} + e^{-2s}\,\mathcal{L}\{3-(t+2)\} + e^{-4s}\,\mathcal{L}\{e^{-3(t+4)}-3\}$

$= \frac{1}{s^2} + e^{-2s}\,\mathcal{L}\{3-t-2\} + e^{-4s}\,\mathcal{L}\{\underbrace{e^{-3t-12}}_{e^{-3t}\cdot e^{-12}}-3\}$

ثابت

~~$= \frac{1}{s^2} + e^{-2s}\left(\frac{1}{s} - \frac{1}{s^2}\right) + e^{-4s}\Big($~~

$$\boxed{= \frac{1}{s^2} + e^{-2s}\left(\frac{1}{s} - \frac{1}{s^2}\right) + e^{-4s}\left(e^{-12}\cdot\frac{1}{s+3} - \frac{3}{s}\right)}$$

Ex) If $g(t) = \begin{cases} t , & 3 < t < 5 \\ 2 , & 5 < t \end{cases}$

find $\mathcal{L}\{g(t)\}$.

soln.

$$g(t) = \begin{cases} 0 , & t < 3 \\ t , & 3 < t < 5 \\ 2 , & 5 < t \end{cases}$$

كي يحقق ع
شرط عند t

$$g(t) = 0 + u(t-3)\cdot(t-0) + u(t-5)\cdot(2-t)$$

$$\mathcal{L}\{g(t)\} = e^{-3s}\,\mathcal{L}\{t+3\} + e^{-5s}\,\mathcal{L}\{2-(t+5)\}$$

$$= e^{-3s}\,\mathcal{L}\{t+3\} + e^{-5s}\,\mathcal{L}\{2-t-5\}$$

$$\boxed{= e^{-3s}\left(\frac{1}{s^2} + \frac{3}{s}\right) + e^{-5s}\left(\frac{-3}{s} - \frac{1}{s^2}\right)}$$

Ex) Evaluate

① $\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s}\right\} = u(t-3)$

② $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^4} - \frac{e^{-4s}}{s^2+9} + \frac{3}{s\,e^{3s}}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^4}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s^2+9}\right\} + 3\,\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s}\right\}$

$= u(t-2)\,\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\}\Big|_{t\to t-2} - u(t-4)\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}\Big|_{t\to t-4}$

$+3u(t-3)$

$= u(t-2)\cdot\frac{t^3}{3}\Big|_{t\to t-2} - u(t-4)\cdot\frac{\sin(3t)}{3}\Big|_{t\to t-4} + 3u(t-3)$

$$\boxed{= u(t-2)\,\frac{(t-2)^3}{3} - u(t-4)\cdot\frac{\sin(3(t-4))}{3} + 3u(t-3)}$$

③ $\mathcal{L}^{-1}\left\{\frac{(s-1)e^{-2s}}{(s-1)^2+16}\right\} = u(t-2)\left[\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+16}\right\}\right]_{t\to t-2}$

$= u(t-2)\left[e^{t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\}\right]_{t\to t-2}$

$= u(t-2)\left[e^{t}\cdot\cos 4t\right]_{t\to t-2}$

$= u(t-2)\left[e^{t-2}\cdot\cos(4(t-2))\right]$

④ $\mathcal{L}^{-1}\left\{\frac{2}{e^{3s}(s-7)^{10}}\right\} = 2\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{(s-7)^{10}}\right\}$

$= 2(u(t-3))\left[\mathcal{L}^{-1}\left\{\frac{1}{(s-7)^{10}}\right\}\right]_{t\to t-3}$

$= 2u(t-3)\left[e^{7t}\mathcal{L}^{-1}\left\{\frac{1}{s^{10}}\right\}\right]_{t\to t-3}$

$= 2u(t-3)\left[e^{7t}\cdot\frac{t^9}{9!}\right]_{t\to t-3}$

$= 2u(t-3)\left[e^{7(t-3)}\cdot\frac{(t-3)^9}{9!}\right]$

H.w) Evaluate

① $\mathcal{L}\{u(t-5)t^2 + u(t-\frac{\pi}{2})\sin t\}$

② $\mathcal{L}^{-1}\{\frac{e^{-3s}}{(s-5)^2+81}\}$

H.w) If $g(t) = \begin{cases} t & , t<2 \\ 0 & , 2\le t<2\pi \\ \cos t & , 2\pi<t \end{cases}$

find $\mathcal{L}\{g(t)\}$

solving the IVP by laplace transformations

Remark:

① $\mathcal{L}\{y^{(n)}(t)\} = s^{n}\mathcal{L}\{y\} - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots\cdots - y^{(n-1)}(0)$

② If $n=2$:

$$\mathcal{L}\{y''(t)\} = s^{2}\mathcal{L}\{y\} - s\,y(0) - y'(0)$$

⑧ If $n=1$:

$$\mathcal{L}\{y'(t)\} = s\,\mathcal{L}\{y\} - y(0)$$

To solve an IVP by laplase transformations:

1- Take the laplace transform for both sides of DE

2- find $\mathcal{L}\{y(t)\} = Y(s)$

~~3- Take~~

3- Take $\mathcal{L}^{-1}$ for both sides in part ②

Ea) Solve the IVP by Laplace Transform:

$$y'' - 6y' + 13y = 0 , \quad y(0) = 0 , \quad y'(0) = -3$$

Soln.: $\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 13\mathcal{L}\{y\} = 0$

$$s^2\mathcal{L}\{y\} - s\,y(0) - y'(0) - 6(s\mathcal{L}\{y\} - y(0)) + 13\mathcal{L}\{y\} =$$

$$= s^2\mathcal{L}\{y\} + 3 - 6s\mathcal{L}\{y\} + 13\mathcal{L}\{y\} = 0$$

$$\mathcal{L}\{y\}\,(s^2 - 6s + 13) = -3$$

$$\boxed{\therefore Y(s) = \mathcal{L}\{y\} = \frac{-3}{s^2-6s+13}}$$

الآن أخذ ال Inverse

$$\Rightarrow y(t) = Y^{-1}(s) = \mathcal{L}^{-1}\left\{\frac{-3}{s^2-6s+13}\right\} \xrightarrow{\text{كمال مربع}}$$

$s^2-6s+13$

$\left(\frac{1}{2}\times \overset{3}{-6}\right)^2 = 9$

$s^2-6s+13+9-9$

$s^2-6s+9+4$

$\boxed{(s-3)^2+4}$

$$= -3\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2+4}\right\}$$

$$= -3e^{3t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= -3e^{3t}\cdot\frac{\sin(2t)}{2} \boxed{= \frac{-3}{2}e^{3t}\sin(2t)}$$

Ex) use laplase transform to solve the IVP

$$y'' - 4y' + 4y = t^3 e^{2t} \quad , \underline{y(0) = 0} \, , \underline{y'(0) = 0}$$

Soln.

$$\mathcal{L}\{y''\} - 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{t^3 e^{2t}\}$$

$$s^2\mathcal{L}\{y\} - sy(0) - y'(0) - 4(s\mathcal{L}\{y\} - y(0)) + 4\mathcal{L}\{y$$

$$= \mathcal{L}\{t^3\}\Big|_{s \to s-2}$$

$$= s^2\mathcal{L}\{y\} - 4s\mathcal{L}\{y\} + 4\mathcal{L}\{y\} = \frac{3!}{s^4}\Big|_{s \to s-2}$$

$$\mathcal{L}\{y\}(s^2 - 4s + 4) = \frac{6}{(s-2)^4}$$

$$\boxed{\therefore Y(s) = \mathcal{L}\{y\} = \frac{6}{(s-2)^4 (s^2 - 4s + 4)}}$$

إذا طلب laplace
بتوقف هون لما اذا طلب ⇐
solution بتكمل

solution ⇒

$$y(t) = Y^{-1}(s) = \mathcal{L}^{-1}\left[\frac{6}{(s-2)^4 (s^2 - 4s + 4)}\right]$$

$$\mathcal{L}^{-1}\left\{\frac{6}{(s-2)^4 (s-2)^2}\right\} \Rightarrow 6\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^6}\right\}$$

$$= 6e^{2t}\mathcal{L}^{-1}\left\{\frac{1}{s^6}\right\} = 6e^{2t}\frac{t^5}{5!}$$

$$\boxed{y(t) = \frac{1}{20}t^5 e^{2t}}$$

Ex) Solve the IVP by laplase transform:

$$y' - 2y = g(t) \quad , y(0) = 0 \; , \; g(t) = \begin{cases} 0 \; , & t < 1 \\ 2 \; , & 1 < t \end{cases}$$

Soln./

~~$g(t) = 0 + u(1-t) \cdot (2-0) = 2u(1-t)$~~

$$g(t) = 0 + u(t-1) \cdot (2-0) = 2u(t-1)$$

$$\therefore \; y' - 2y = 2u(t-1)$$

$$\mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 2\mathcal{L}\{u(t-1)\}$$

$$s\mathcal{L}\{y\} - \cancel{y(0)}^{0} - 2\mathcal{L}\{y\} = 2\frac{e^{-s}}{s}$$

$$\mathcal{L}\{y\}(s-2) = 2\frac{e^{-s}}{s}$$

$$Y(s) = \mathcal{L}\{y\} = 2\frac{e^{-s}}{s(s-2)}$$

soln. $\Rightarrow \; y(t) = \mathcal{L}^{-1}Y(s) = 2\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s-2)}\right\}$

$$= 2u(t-1)\left[\mathcal{L}^{-1}\left\{\frac{1}{(s-2)s}\right\}\right]_{t \to t-1}$$

كسور جزئية

$$= 2u(t-1)\left[\mathcal{L}^{-1}\left\{\frac{\frac{1}{2}}{s-2} + \frac{-\frac{1}{2}}{s}\right\}\right]_{t \to t-1}$$

~~$A(s-2)$~~

$$= 2u(t-1)\left[\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}\right]_{t \to t-1}$$

$$1 = A(s) + B(s-2)$$

$$\boxed{s=0} \Rightarrow \boxed{B = \frac{-1}{2}}$$

$$\boxed{s=2} \Rightarrow \boxed{A = \frac{1}{2}}$$

$$= 2u(t-1)\left[\frac{1}{2}e^{2t} - \frac{1}{2}\cdot 1\right]_{t \to t-1} \longrightarrow$$

$$= 2\,u(t-1)\left[\frac{1}{2}e^{2(t-1)} - \frac{1}{2}\right]$$

$$\boxed{= u(t-1)\left(e^{2t-2} - 1\right)}$$

H.w: use laplase transform to solve the IVP:

① $y'' + 2y' + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$

② $y' + y = f(t)$, $y(0) = 0$, $f(t) = \begin{cases} 0 & t < 2 \\ 5 & 2 < t \end{cases}$

③ $y'' + ty' - 2y = -2$, $y(0) = 0$, $y'(0) = 0$

Soln $\mathcal{L}\{y''\} + \mathcal{L}\{ty'\} - 2\mathcal{L}\{y\} = \mathcal{L}\{-2\}$

$$s^2\mathcal{L}\{y\} - s\,y(0) - y'(0) + -\left[\mathcal{L}\{y'\}\right]' - 2\mathcal{L}\{y\} =$$

$$\Rightarrow s^2\mathcal{L}\{y\} - \left[s\mathcal{L}\{y\} - y(0)\right]' - 2\mathcal{L}\{y\} = \frac{-2}{s}$$

take $\mathcal{L}\{y\} = Y(s)$

$\mathcal{L}\{t^n f(t)\} = (-1)^n\left[\mathcal{L}\{f(t)\}\right]^{(n)}$

$n = 1$

$\mathcal{L}\{t f(t)\} = -\left[\mathcal{L}\{f(t)\}\right]'$

$$s^2 Y(s) - \left[s\,Y(s)\right]' - 2Y(s) = \frac{-2}{s}$$

$$s^2 Y(s) - Y(s) - s\,Y'(s) - 2Y(s) = \frac{-2}{s}$$

$$(s^2 - 3)Y(s) - s\,Y'(s) = \frac{-2}{s} \Rightarrow -sY' + (s^2 + 3)Y(s) = \frac{-2}{s}$$

$$\Rightarrow Y'(s) + \frac{(s^2-3)}{-s}Y(s) = \frac{2}{s^2} \quad \text{-- linear in } Y$$

$P(s) = \frac{s^2 - 3}{} \boxed{= -s + \frac{3}{}}$, $q(s) = \frac{-2}{}$

$$\Rightarrow \mu(s) = e^{\int P(s)\, ds} = e^{\int (-s + \frac{3}{s})\, ds} = e^{-\frac{s^2}{2} + 3\ln s}$$

$$= e^{-\frac{s^2}{2} + \ln s^3}$$

$$= e^{-\frac{s^2}{2}} \cdot e^{\ln s^3}$$

$$= e^{-s^2/2} \cdot s^3$$

$$\boxed{= s^3 e^{-s^2/2}}$$

$$\therefore Y = \frac{1}{\mu(s)} \int \mu(s)\, g(s)\, ds + \frac{c}{\mu(s)}$$

$$\cancel{Y = \frac{e^{s^2/2}}{s^3}} \quad Y = \frac{e^{s^2/2}}{s^3} \int s^3 e^{-\frac{s^2}{2}} \cdot \frac{2}{s^2}\, ds + \frac{c}{s^3 \cdot e^{-s^2/2}}$$

$$= \frac{e^{\frac{s^2}{2}}}{s^3} \int \underbrace{2 s e^{-\frac{s^2}{2}}}_{\text{بالتعويض}} ds + c \cdot \frac{e^{s^2/2}}{s^3}$$

$$= \frac{e^{\frac{s^2}{2}}}{s^3} \cdot \left[-2 e^{-\frac{s^2}{2}}\right] + c \cdot \frac{e^{s^2/2}}{s^3}$$

$$Y(s) = \frac{-2}{s^3} + c \cdot \frac{e^{\frac{s^2}{2}}}{s^3}$$

$$\lim_{s \to \infty} Y(s) = 0$$

$$\therefore c = 0$$

$$\therefore Y(s) = \frac{-2}{s^3}$$

soln.

$$\Rightarrow y(t) = Y^{-1}(s) = \mathcal{L}^{-1}\left\{\frac{-2}{s^3}\right\}$$

$$= -2 \mathcal{L}^{-1}\left[\frac{1}{s^3}\right] = -2 \frac{t^2}{2!} = \frac{-2t^2}{2}$$

$$\boxed{y(t) = -t^2}$$

مش لأنها $c=0$ فقط كي يتحقق الشرط

لأن $\lim_{s \to \infty} \frac{e^{\frac{s^2}{2}}}{s^3} \neq 0$ إذا $c \neq 0$